Further Triangle tilings

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Abstract

Here we give a more complete reckoning of the conjecture discussed in "Regular Production Systems and Triangle Tilings" [16]

Here we discuss which triangles do, and which don't, admit a tiling of $\mathbb{H}^2, \mathbb{E}^2$, and $\mathbb{S}^2$. These notes are meant to pick up as "Regular Production Systems and Triangle Tilings" [16] leaves off. We pause for a conjecture and some additional nomenclature:

Let T be any triangle in $X = \mathbb{H}^2, \mathbb{E}^2, \mathbb{S}^2$. A **configuration** by T is a collection of congruent copies of T, for each pair of which meet edge-to-edge and vertex-to-vertex (that is, each pair has disjoint interiors and the vertex of one is not in the interior of the edge of another). A **tiling** by T is a configuration that covers all of X. A triangle **admits a tiling** iff there exists such a tiling.

A **vertex arrangement** of T is a configuration by T all meeting meeting at and surrounding a point. Inductively, we define ***n*-nions** ("onions") of T: a 0-nion is a vertex arrangment by T. A n-nion is any configuration C by T that (a) contains an $(n-1)-nion$ C' in the interior of C and (b) every copy of T is either in C' or incident to both C' and the boundary of C. That is, an n-nion really is a kind of onion, the union concentric layers of triangles.

We discuss:

Conjecture 0.1 *There exists a (very small) N such that: for all triangles T, T admits a tiling if and only there exists a N-nion by T.*

1 A little notation for cases

To simplify our analysis and conflate cases we adopt the following notation. A "form" $|F|$ denotes a family of sets of planes in Π; these families will be invariant under the symmetry of $\mathcal{O}$. Unfortunately, it will be possible to denote a given family in many ways.

A form will consist of rows of three symbols each. The symbols are $*, d, e, 0/1$, standing respectively for any natural number, any odd natural, any even natural, and either 0 or 1; and for any fixed natural numbers n, m we also have n, n^+, n^{+m} standing for n, any $n+k$, any $n+2k$, any $n+mk$, respectively, with $k \geq 0$. In addition a row may be demarked by $\natural$.

A plane π_{rst} is of the form of a given row xyz if and only if x, y, z can stand for r, s, t. A set $P \subset \Pi$ is an element of a form F if and only if: There exists a Q (possibly P itself) such that (a) $Q = \rho P$ for some permutation ρ of the coordinates of $\mathcal{O}$; (b) every plane in Q is of the form of some row of F; and (c) every non-demarked row is the form of exactly one plane in Q (demarked forms may be represented once, or more, or not at all, in Q).

So for example, letting $P = \{\pi_{241}, \pi_{357}\}$, $P \in \begin{vmatrix} e & 2^+ & 1 \\ 3^+ & 3^+ & 1^{+3} \end{vmatrix}$, among many other forms.

Note we can denote a family of sets of planes in many ways; for example: $\begin{vmatrix} 2 & 6 & 3 \\ 0 & 8 & 6 \end{vmatrix} = \begin{vmatrix} 8 & 6 & 0 \\ 6 & 3 & 2 \end{vmatrix}$.

Given a form $|F|$, we will write

$$[F] := \text{"for all } P \in |F|, [P]\text{"}$$

$$(F) := \text{"for all } P \in |F|, (P)\text{"}$$

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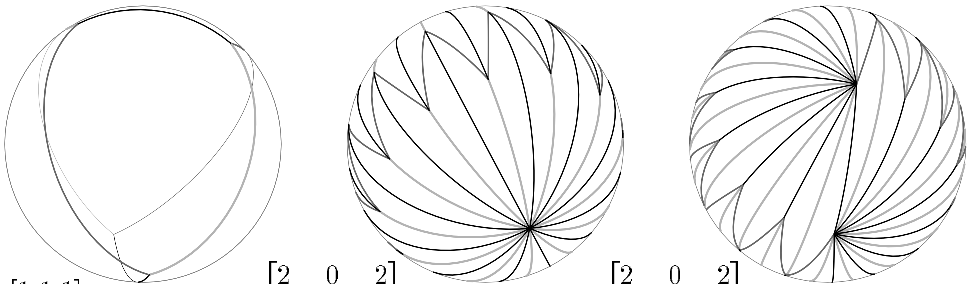
We will also take $\mathcal{V}(F) := \{\mathcal{V}(P)\}$. Recall that the notation was chosen so that the statements $[P], (P)$ refer to tilings by $\mathsf{T} \in \tilde{P}$; consequently, it is perfectly possible that (F) does not hold even though there is no 2-nion, say, by any $V \in \mathcal{V}(F)$. This device allows us to distinguish between the statements “there is a tiling using only such-and-such vertex arrangements” and “there is a tiling by triangles whose angles satisfy such-and-such system of equations”. Lemma 1.1 helps a great deal as we analyze cases later.



$[1\ 1\ 1]$ $\begin{bmatrix}2 & 0 & 2\\ 0 & 2^+ & 0\end{bmatrix}$ $\begin{bmatrix}2 & 0 & 2\\ 1 & 3^+ & 1\end{bmatrix}$

Lemma 1.1 *1.* $(\natural\ 0/1\ *\ *)$, *with the exception of the scalene tetrahedra* $[1\ 1\ 1]$.

2. $\begin{pmatrix}\natural & 2 & 0 & 2^+\\ \natural & 0/1 & * & *\end{pmatrix}$ *with the exceptions of* $\begin{bmatrix}2 & 0 & 2\\ 0 & 2^+ & 0\end{bmatrix}$, $\begin{bmatrix}2 & 0 & 2\\ 1 & 3^+ & 1\end{bmatrix}$.

3. $\begin{pmatrix}2 & 0 & 0\\ \natural* & * & *\end{pmatrix}$ *with the exception that for all non-negative p, positive q:* $\begin{bmatrix}2 & 0 & 0\\ 0 & 2p & 2q\\ 0 & 2q & 2p\end{bmatrix}$ *and* $\begin{bmatrix}2 & 0 & 0\\ 1 & 2p+1 & 2q+1\\ 1 & 2q+1 & 2p+1\end{bmatrix}$.

4. $\begin{pmatrix}\natural & 2^{+4} & 4 & 0\\ \natural & 4^{+4} & 0 & 4\end{pmatrix}$.

The exceptional cases 1 and 2 are illustrated in Figure 1. The remaining exceptions are all tilings by lune-shaped triangles, with one vertex angle π, or for $\begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}$, the tiling by two hemispherical triangles, with vertex angles all π.

Proof The real point of the theorem, or at least the proof, is that if $\mathcal{V}$ is a set of vertex arrangements in any of $\mathcal{V}(\natural\ 0/1\ *\ *)$, $\nu\begin{pmatrix}\natural & 2 & 0 & e\\ \natural & 0/1 & * & *\end{pmatrix}$, or $\nu\begin{pmatrix}\natural & 0^+ & 4 & 0\\ \natural & 0^+ & 0 & 4\end{pmatrix}$, then we can show there is no 2-nion formed using arrangements in $\mathcal{V}$, unless $\mathcal{V}$ is one of the stated exceptions. (In fact, many of the exceptions admit no 2-nion either, since they give rise to rather small spherical tilings!). We will prove only one case below, after outlining the consequences; the other cases differ only in tedium.

Now if P is in $|\natural\ 0/1\ *\ *|$, $\begin{vmatrix}\natural & 2 & 0 & e\\ \natural & 0/1 & * & *\end{vmatrix}$, or $\begin{vmatrix}\natural & 2^{+4} & 4 & 0\\ \natural & 0^{+4} & 0 & 4\end{vmatrix}$, so too is $\tilde{P}$, and so we can write (P), with the stated exceptions. However, we do have to leave the form $F = \begin{vmatrix}4^{+4} & 4 & 0\\ 4^{+4} & 0 & 4\end{vmatrix}$ open; if $P \in F$, there is a plane in $|2^+\ 2\ 2| \cap \tilde{P}$. In Proposition 3.3 we will see $[2^+22]$, hence $\left[\tilde{P}\right]$ and $[P]$.

In any case, we will prove the initial assertion, that there is no 2-nion by vertices in $\mathcal{V}(F)$, in the case that $F = |\natural\ 0/1\ *\ *|$. Let $P \in F$ and $\mathsf{T} \in \mathring{\cap} P$. Clearly if for all $\pi_{rst} \in P$, we have $r = 0$, then there can be no vertex arrangements containing the vertex α, and so T does not admit any 1-nion, much less a tiling. Suppose then in any vertex arrangement of T there is at most one copy of the vertex α. By examing the graph Γ_{T}, we see that if we do not have $r = s = t = 1$, our vertex arrangment must have a consecutive pair $\overline{\beta}\beta$ or $\gamma\overline{\gamma}$. But then to complete a 1-nion, we then require a vertex arrangment with two copies of α. Therefore, no such triangle admits a 2-nion or a tiling. The other cases go in similar fashion. ☐

2 Topological Methods

The following trick is helpful. Essentially this amounts to a move on sets of vertex arrangements. For a given set of planes $Q = \{\pi_{rst}\}$, let Q^{+1} denote the set of planes $\{\pi_{(r+1)(s+1)(t+1)}\}$— that is we increment each index of each plane by 1.

Lemma 2.1 *Let $P, Q \subset \Pi$ and let $\mathcal{V} = \mathcal{V}(P \cup Q)$, $\mathcal{V}' = \mathcal{V}(P \cup Q^{+})$. Suppose there is a $\mathcal{V}$-complex $\mathcal{C}$, with a collection of edges $\mathcal{E}$ in $\mathcal{C}$ such that every vertex of $\mathcal{C}$ of the form of one of $\mathcal{V}(Q)$ is incident to exactly one edge in $\mathcal{E}$, and every edge of $\mathcal{E}$ has ends on vertices of the form $\mathcal{V}(Q)$. Then there exists a $\mathcal{V}'$-complex; indeed, if $|\mathcal{E}| \geq 2$, there exist uncountably many non-homeomorphic $\mathcal{V}'$-complices.*

Proof The proof is rather simple, in outline. A very careful proof, especially of the last statement of the theorem, is somewhat more tedious.

We'll first construct one $\mathcal{V}'$-complex. Replace every edge in $\mathcal{E}$ with a lune as shown below:

The orientation of the lune does not matter particularly. The three labels x, y, z are A, B, C (or their reflections, depending on the orientation), but we are free to assign which is which so long as the labels on the outside edges are compatible with the edge in $\mathcal{E}$.

Now simply take the universal cover of the two-fold branched cover through all the marked points in the interior of all the lunes. The resulting complex will be a $\mathcal{V}'$-complex. Each lune will become a strip as shown at right above.

But in fact, there are far more choices available when $|\mathcal{E}| \geq 2$. Let $\mathcal{C}_1$ be the complex formed from $\mathcal{C}$ by doubling each edge of $\mathcal{E}$, creating a lune shaped hole bounded by two copies of the original edge. Let $\mathcal{C}_2 = \tilde{\mathcal{C}_1}$ be the universal cover of $\mathcal{C}_1$. $\mathcal{C}_2$ now has either two, or infinitely many, boundary components. Each component may be marked A, B, C depending on the markings of the original edge in $\mathcal{E}$. Define three strips s_A, s_B, s_C of triangles: s_x is shown above.

We will not make this next bit completely precise; to do so only requires a careful induction. Begin with a copy of $\mathcal{C}_2$ and to each boundary component labeled $X = A, B, C$, attach a copy of s_X, making an arbitrary choice of orientation. Next, to each boundary component, adjoining some s_X, $X = A, B, C$ of this new complex attach a copy of $\mathcal{C}_2$, along a boundary labeled X, again with an arbitrary choice of orientation. At each stage, repeat this process, ad infinitum.

Now note that every vertex in the final complex is of the form of one of $\mathcal{V}'$ and we have a $\mathcal{V}'$-complex. □

We give a quick application or two:

Note that in this way, $\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2p+1 & 2q+1 \\ 1 & 2q+1 & 2p+1 \end{bmatrix}$ can be produced from $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2p & 2q \\ 0 & 2q & 2p \end{bmatrix}$.

Similarly, by choosing appropriate edges in $\begin{bmatrix} 2 & 0 & 2 \\ 0 & 2^{+} & 0 \end{bmatrix}$ we have:

Corollary 2.2 $\begin{bmatrix} 3 & 3 & 1 \\ 0 & 0 & 2^{+} \end{bmatrix}$ *and* $\begin{bmatrix} 3 & 3 & 1 \\ 1 & 1 & 3^{+} \end{bmatrix}$

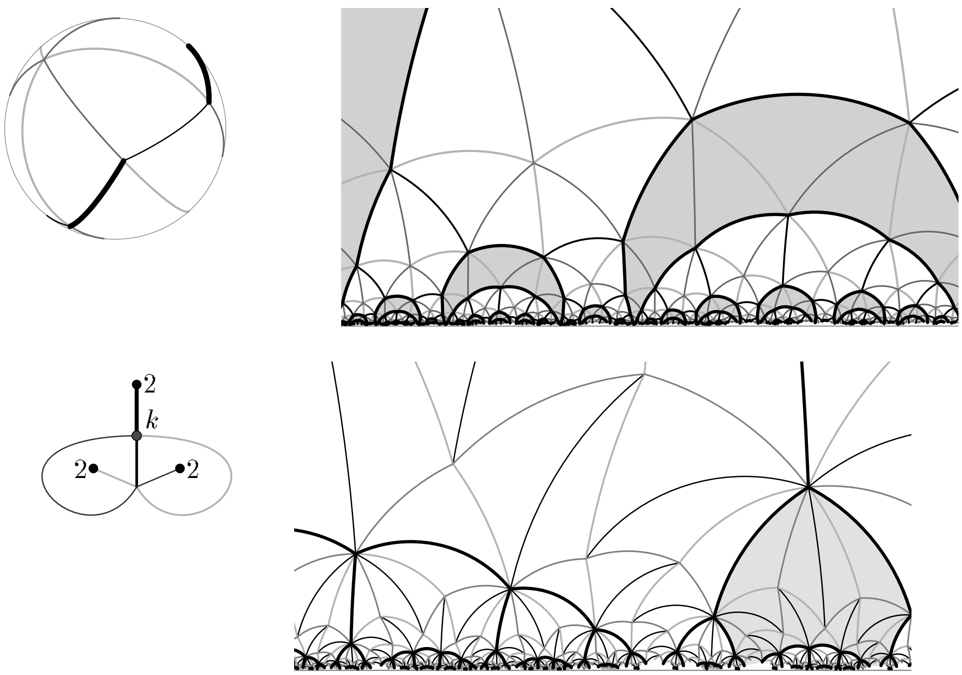
We can use branched covers in other ways as well:

Theorem 2.3 *For all m, n: $\begin{pmatrix} n & m & 0 \\ 1 & 1 & 3 \end{pmatrix}$, with the exception that for all even $n \geq 4$, $\begin{bmatrix} n & n & 0 \\ 1 & 1 & 3 \end{bmatrix}$*

Proof First, let n be even and positive. Then letting $\mathcal{V} = \nu \begin{pmatrix} n & n & 0 \\ 1 & 1 & 3 \end{pmatrix}$, construct a $\mathcal{V}$-complex by taking the universal cover of the indicated branched cover of the graph in Figure 2.

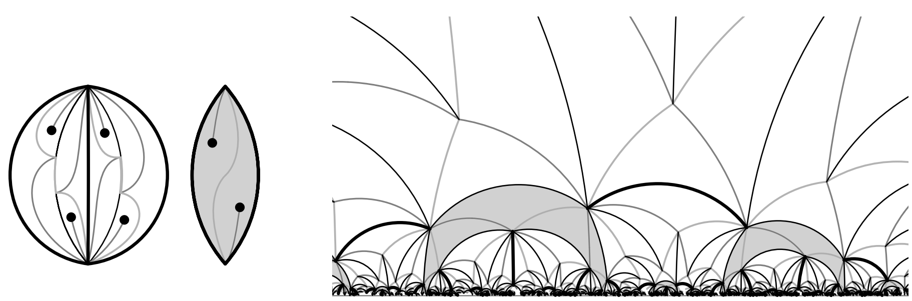
On the other hand, it is not difficult to show that there are no 2-nions using vertex arrangements in $\nu \begin{pmatrix} n & m & 0 \\ 1 & 1 & 3 \end{pmatrix}$, $m \neq n$. □

We close this section with another illustration of Lemma 2.1.



Theorem 2.4 *For all k:* $\begin{bmatrix} 2k+1 & 2k+1 & 1 \\ 1 & 1 & 3 \end{bmatrix}$

Proof We give one such tiling; it will be clear there are uncountably many. Consider the graph and orbifold at left in Figure 2. Take the branched cover through all but the lower left branch point; the result consists of a sphere decorated with k copies of the lune illustrated at left in Figure 2. The universal cover of the 2-fold branched cover through the marked points is a tiling T in $\begin{bmatrix} 2k & 2k & 0 \\ 1 & 1 & 3 \end{bmatrix}$. Now select any one edge in the graph running alongside one lune. The preimage of this edge in T is a collection of edges $\mathcal{E}$ in T satisfying the conditions of Lemma 2.1 with $Q \in \mathcal{V}(2k\ 2k\ 0)$. That is, $\begin{bmatrix} 2k+1 & 2k+1 & 1 \\ 1 & 1 & 3 \end{bmatrix}$. □



3 Theorems

Before delving into the combinatorial thicket of the space of triangles, we pause to illustrate these techniques with a well-known theorem:

3.1 The Poincaré triangle theorem

Theorem 3.1 (Poincaré) $\begin{bmatrix} 4^+ & 0 & 0 \\ 0 & 4^+ & 0 \\ 0 & 0 & 4^+ \end{bmatrix}$.

In fact we only prove here $\begin{bmatrix} 6^+ & 0 & 0 \\ 0 & 6^+ & 0 \\ 0 & 0 & 6^+ \end{bmatrix}$.

Proof Let $P = \{\pi_{(2p)00}, \pi_{0(2q)0}, \pi_{00(2r)}$ and let $\mathsf{T} \in \cap P$. Let $\mathcal{V} = \mathcal{V}(P)$. Then certainly the three vertex configurations $(\alpha\overline{\alpha})^p, (\beta\overline{\beta})^q, (\gamma\overline{\gamma})^r$ lie in $\mathcal{V}$ and the following rules lie in $\mathcal{R}_{\mathcal{V}}$:

$(\alpha\overline{\alpha})^p$	$(\beta\overline{\beta})^q$	$(\gamma\overline{\gamma})^r$
$[B\overline{B}] \mapsto (A\overline{A})^{i+1}\beta\overline{C}$	$[C\overline{C}] \mapsto (B\overline{B})^{j+1}\gamma\overline{A}$	$[A\overline{A}] \mapsto (C\overline{C})^{k+1}\alpha\overline{B}$
$[\overline{C}C] \mapsto (\overline{A}A)^{i+1}\overline{\gamma}B$	$[\overline{A}A] \mapsto (\overline{B}B)^{j+1}\overline{\alpha}C$	$[\overline{B}B] \mapsto (\overline{C}C)^{k+1}\overline{\alpha}A$
$[B\overline{\alpha}C] \mapsto A(\overline{A}A)^i\overline{\gamma}B$	$[C\overline{\beta}A] \mapsto B(\overline{B}B)^j\overline{\alpha}C$	$[A\overline{\gamma}B] \mapsto C(\overline{C}C)^k\overline{\beta}A$
$[\overline{C}\alpha\overline{B}] \mapsto \overline{A}(A\overline{A})^i\beta\overline{C}$	$[\overline{A}\beta\overline{C}] \mapsto \overline{B}(B\overline{B})^j\gamma\overline{A}$	$[\overline{B}\gamma\overline{A}] \mapsto \overline{C}(C\overline{C})^k\alpha\overline{B}$

where $i = p - 3$, $j = q - 3$, $k = r - 3$; recall our convention for abbreviating the words in $\mathcal{L}$. For example, $(A\overline{A})^2\beta\overline{C} = A\overline{A}A\overline{A}\beta\overline{C} = [A\overline{A}][\overline{A}A][A\overline{A}][\overline{A}\beta\overline{C}]$.

Then taking $\mathcal{A}^-$ to be the twelve letters on the left of each rule, taking $\mathcal{L}^- = \mathcal{L}|_{\mathcal{A}^-}$ (that is, $\mathcal{L}$ restricted to the letters of $\mathcal{A}^-$) and $\mathcal{R}^-$ to be specified by the twelve rules above, one can easily check that $(\mathcal{A}^-, \mathcal{L}^-, \mathcal{R}^-)$ forms a cut of $(\mathcal{A}, \mathcal{L}, \mathcal{R}_{\mathcal{V}})$. Consequently, there is a orbit in $\mathcal{L}^\infty$ under $\mathcal{R}$, there is a $\mathcal{V}$-complex, and finally, T admits a tiling. ☐

Indeed, in some ways, this is precisely the core of Poincaré's construction.

3.2 A quick lemma

The following lemma is quite helpful and generalizes readily.

Lemma 3.2 *Let $\mathcal{V}$ be a set of vertex arrangements. Then $(\mathcal{A}, \mathcal{L}, \mathcal{R}_{\mathcal{V}})$ has a cut if there exists a function $f : \{A, B, C, \overline{A}, \overline{B}, \overline{C}\} \to \{A, B, C, \overline{A}, \overline{B}, \overline{C}\}$ such that there is a non-empty collection of letters $\mathcal{A}^- \subset \mathcal{A}$ and rules $\mathcal{R}^- \subset \mathcal{R}_{\mathcal{V}}$, one or more for each letter in $\mathcal{A}^-$, each of the form $[X \ldots Y] \overset{\mathcal{R}}{\mapsto} [f(X) \ldots] \ldots [\ldots f(Y)]$ with letters in $\mathcal{A}^-$.*

Proof This is a trivial application of Lemma 2.9 of [16], taking one rule for each letter. ☐

3.3 On 2-dimensional affine subspaces in $\cup\Pi$

Recalling that (dde), (dee), $(0/1\ \ **)$ with the exception of [111], this Proposition completely settles the question of whether or not a given triangle, lying on exactly one plane of Π, admits a tiling. That is, we now have given necessary and sufficient conditions for admitting a tiling, for a measure-one set of triangles in $\cup\Pi$.

Proposition 3.3 $[2^+\ 2^+\ 2^+]$ *and* $[3^+\ 3^+\ 3^+]$

Please note that in throughout the remainder of the paper we will be content to find *one* cut or lift; but the particular choice we make is rather arbitrary.

Proof We simply demonstrate that one may take the identity function for f in an application of Lemma 3.2. In each case we'll simply list the rules in $\mathcal{R}^-$; the alphabet $\mathcal{A}^-$ will be given as the letters on the left of the rules, and of course $\mathcal{L}^-$ just is $\mathcal{L}_T$ restricted to $\mathcal{A}^-$. For the most active reader, we list, to the right of the rules, the vertex arrangements that produce each rule; these are given by listing the ζ_i's.

Finally, note that the words in $\mathcal{L}^-$ are abbreviated, as we discuss above. Hence $[A\overline{A}] \to (A\overline{A})^i A\overline{A}\overline{\alpha}\overline{A}$ should be read as $[A\overline{A}] \to ([A\overline{A}][\overline{A}A])^i[A\overline{A}][\overline{A}\overline{\alpha}\overline{A}]$.

Claim: $[(2 + 2i)\ 2\ 2]$, $i >= 0$.

$[A\overline{A}] \to (A\overline{A})^i A\overline{A}\overline{\alpha}\overline{A}$	$\gamma\overline{\gamma}\overline{\beta}\overline{\alpha}\alpha(\overline{\alpha}\alpha)^i\beta$
$[\overline{A}A] \to (\overline{A}A)^i\overline{A}A\alpha A$	$\overline{\beta}\beta\gamma\alpha\overline{\alpha}(\alpha\overline{\alpha})^i\overline{\gamma}$
$[A\alpha A] \to (A\overline{A})^i A\alpha A$	$\gamma\alpha\beta\gamma\alpha(\overline{\alpha}\alpha)^i\beta$
$[\overline{A}\overline{\alpha}\overline{A}] \to (\overline{A}A)^i\overline{A}\overline{\alpha}\overline{A}$	$\overline{\beta}\overline{\alpha}\overline{\gamma}\overline{\beta}\overline{\alpha}(\alpha\overline{\alpha})^i\overline{\gamma}$

Claim: $[(2+2i)\ (2+2j)\ 2]$, $i, j >= 0$

$$
\begin{array}{rlr}
[A\overline{A}] & \to (A\overline{A})^i(\overline{B}B)^j A\overline{A}\overline{\alpha}\overline{A} & \gamma\overline{\gamma}\overline{\beta}\overline{\alpha}\alpha(\beta\overline{\beta})^j(\overline{\alpha}\alpha)^i\beta \\
[BA] & \to B(A\overline{A})^i(\overline{B}B)^j A\alpha A & \alpha\beta\gamma\alpha(\beta\overline{\beta})^j(\overline{\alpha}\alpha)^i\beta\gamma \\
[B\overline{B}] & \to B(A\overline{A})^i(\overline{B}B)^j \overline{B\beta B} & \alpha\overline{\alpha\gamma}\overline{\beta}(\beta\overline{\beta})^j(\overline{\alpha}\alpha)^i\beta\gamma \\
[\overline{A}A] & \to \overline{A}(A\overline{A})^i(\overline{B}B)^j A\alpha A & \overline{\beta}\beta\gamma\alpha(\beta\overline{\beta})^j(\overline{\alpha}\alpha)^i\overline{\alpha\gamma} \\
[\overline{AB}] & \to \overline{A}(A\overline{A})^i(\overline{B}B)^j \overline{B\beta B} & \overline{\beta}\overline{\alpha\gamma}\overline{\beta}(\beta\overline{\beta})^j(\overline{\alpha}\alpha)^i\overline{\alpha\gamma} \\
[\overline{B}B] & \to (\overline{B}B)^j(A\overline{A})^i\overline{B}B\beta B & \overline{\gamma}\gamma\alpha\beta\overline{\beta}(\overline{\alpha}\alpha)^i(\beta\overline{\beta})^j\overline{\alpha} \\
[A\alpha A] & \to (A\overline{A})^i(\overline{B}B)^j A\alpha A & \gamma\alpha\beta\gamma\alpha(\beta\overline{\beta})^j(\overline{\alpha}\alpha)^i\beta \\
[B\beta B] & \to B(A\overline{A})^i(\overline{B}B)^j \beta B & \alpha\beta\gamma\alpha(\beta\overline{\beta})^j(\overline{\alpha}\alpha)^i\beta\gamma \\
[\overline{A\alpha A}] & \to \overline{A}(\overline{B}B)^j(A\overline{A})^i\overline{\alpha A} & \overline{\beta}\overline{\alpha\gamma}\overline{\beta}(\overline{\alpha}\alpha)^i(\beta\overline{\beta})^j\overline{\alpha\gamma} \\
[\overline{B\beta B}] & \to (\overline{B}B)^j(A\overline{A})^i\overline{B\beta B} & \overline{\gamma}\overline{\beta}\overline{\alpha\gamma}\overline{\beta}(\overline{\alpha}\alpha)^i(\beta\overline{\beta})^j\overline{\alpha}
\end{array}
$$

Claim: $[(2+2i+\delta)\ (2+2j+\delta)\ (2+2k+\delta)]$ with $i=j=k=\delta=0$ or $i,j,k \geq 0, \delta = 1, 2$.

$$
\begin{array}{rlll rlll}
[AC] & \to A^\rho C\gamma C & \gamma\alpha\beta\gamma\alpha^\rho\beta & \quad & [A\overline{A}] & \to A^\rho\overline{A}\overline{\alpha}\overline{A} & \gamma\overline{\gamma}\overline{\beta}\overline{\alpha}\alpha^\rho\beta \\
[BA] & \to B^\rho A\alpha A & \alpha\beta\gamma\alpha\beta^\rho\gamma & & [B\overline{B}] & \to B^\rho\overline{B\beta B} & \alpha\overline{\alpha\gamma}\overline{\beta}\beta^\rho\gamma \\
[CB] & \to C^\rho B\beta B & \beta\gamma\alpha\beta\gamma^\rho\alpha & & [C\overline{C}] & \to C^\rho\overline{C}\overline{\gamma}\overline{C} & \beta\overline{\beta}\overline{\alpha\gamma}\gamma^\rho\alpha \\
[\overline{A}A] & \to \overline{A}^\rho A\alpha A & \overline{\beta}\beta\gamma\alpha\overline{\alpha}^\rho\overline{\gamma} & & [\overline{AB}] & \to \overline{A}^\rho\overline{B\beta B} & \overline{\beta}\overline{\alpha\gamma}\overline{\beta}\overline{\alpha}^\rho\overline{\gamma} \\
[\overline{B}B] & \to \overline{B}^\rho B\beta B & \overline{\gamma}\gamma\alpha\beta\overline{\beta}^\rho\overline{\alpha} & & [\overline{BC}] & \to \overline{B}^\rho\overline{C}\overline{\gamma}\overline{C} & \overline{\gamma}\overline{\beta}\overline{\alpha\gamma}\overline{\beta}^\rho\overline{\alpha} \\
[\overline{C}C] & \to \overline{C}^\rho C\gamma C & \overline{\alpha}\alpha\beta\gamma\overline{\gamma}^\rho\overline{\beta} & & [\overline{CA}] & \to \overline{C}^\rho\overline{A}\overline{\alpha}\overline{A} & \overline{\alpha\gamma}\overline{\beta}\overline{\alpha\gamma\gamma}^\rho\overline{\beta} \\
[A\alpha A] & \to A^\rho\alpha A & \gamma\alpha\beta\gamma\alpha^\rho\beta & & [B\beta B] & \to B^\rho\beta B & \alpha\beta\gamma\alpha\beta^\rho\gamma \\
[C\gamma C] & \to C^\rho\gamma C & \beta\gamma\alpha\beta\gamma^\rho\alpha & & [\overline{A\alpha A}] & \to \overline{A}^\rho\overline{\alpha}\overline{A} & \overline{\beta}\overline{\alpha\gamma}\overline{\beta}\overline{\alpha}^\rho\overline{\gamma} \\
[\overline{B\beta B}] & \to \overline{B}^\rho\overline{\beta B} & \overline{\gamma}\overline{\beta}\overline{\alpha\gamma}\overline{\beta}^\rho\overline{\alpha} & & [\overline{C\gamma C}] & \to \overline{C}^\rho\overline{\gamma}\overline{C} & \overline{\alpha\gamma}\overline{\beta}\overline{\alpha\gamma\gamma}^\rho\overline{\beta}
\end{array}
$$

where if $\delta = 0$, $x^\rho = x$ for all x; and if $\delta = 1, 2$, taking $l = 2 - \delta$,

$$
\begin{array}{ll}
A^\rho = A(CBA)^l C(\overline{C}C)^k B(\overline{B}B)^j A(\overline{A}A)^i & \alpha^\rho = (\alpha\overline{\alpha})^i\alpha(\beta\overline{\beta})^j\beta(\gamma\overline{\gamma})^k\gamma(\alpha\beta\gamma)^l\alpha \\
B^\rho = B(ACB)^l A(\overline{A}A)^i C(\overline{C}C)^k B(\overline{B}B)^j & \beta^\rho = (\beta\overline{\beta})^j\beta(\gamma\overline{\gamma})^k\gamma(\alpha\overline{\alpha})^i\alpha(\beta\gamma\alpha)^l\beta \\
C^\rho = C(BAC)^l B(\overline{B}B)^j A(\overline{A}A)^i C(\overline{C}C)^k & \gamma^\rho = (\gamma\overline{\gamma})^k\gamma(\alpha\overline{\alpha})^i\alpha(\beta\overline{\beta})^j\beta(\gamma\alpha\beta)^l\gamma \\
\overline{A}^\rho = \overline{A}(\overline{BCA})^l\overline{B}(B\overline{B})^j\overline{C}(C\overline{C})^k\overline{A}(A\overline{A})^i & \overline{\alpha}^\rho = (\overline{\alpha}\alpha)^i\overline{\alpha}(\overline{\gamma}\gamma)^k\overline{\gamma}(\overline{\beta}\beta)^j\overline{\beta}(\overline{\alpha\gamma\beta})^l\overline{\alpha} \\
\overline{B}^\rho = \overline{B}(\overline{CAB})^l\overline{C}(C\overline{C})^k\overline{A}(A\overline{A})^i\overline{B}(B\overline{B})^j & \overline{\beta}^\rho = (\overline{\beta}\beta)^j\overline{\beta}(\overline{\alpha}\alpha)^i\overline{\alpha}(\overline{\gamma}\gamma)^k\overline{\gamma}(\overline{\beta\alpha\gamma})^l\overline{\beta} \\
\overline{C}^\rho = \overline{C}(\overline{ABC})^l\overline{A}(A\overline{A})^i\overline{B}(B\overline{B})^j\overline{C}(C\overline{C})^k & \overline{\gamma}^\rho = (\overline{\gamma}\gamma)^k\overline{\gamma}(\overline{\beta}\beta)^j\overline{\beta}(\overline{\alpha}\alpha)^i\overline{\alpha}(\overline{\gamma\beta\alpha})^l\overline{\gamma}
\end{array}
$$

Essentially we are inserting cycles of the graph of Figure 7 of [16] into the vertices to expand out the rules as needed. □

3.4 On 1-dimensional affine subspaces in $\cup\Pi$

With Lemma 1.1, the lemmas of Section 3.2 of [16] and Proposition 3.3, we can now discuss the landscape of the remaining cases. The general forms that have non-empty vertex arrangements are: *ddd* and *eee*. We now know that [111], $[2^+2^+2^+]$, $[3^+3^+3^+]$ and $(0{*}{*})$, $(1{*}{*})$; as we turn to triangles lying on exactly two independent planes, we are left with the cases:

$$
\begin{vmatrix} 3^+ & 3^+ & 1 \\ 3^+ & 1 & 3^+ \end{vmatrix} \qquad
\begin{vmatrix} 3^+ & 3^+ & 1 \\ 1 & 1 & 3^+ \end{vmatrix} \qquad
\begin{vmatrix} 3^+ & 3^+ & 1 \\ 2^+ & 0 & 2^+ \end{vmatrix} \qquad
\begin{vmatrix} 3^+ & 3^+ & 1 \\ 0 & 0 & 2^+ \end{vmatrix}
$$

$$
\begin{vmatrix} 3^+ & 1 & 1 \\ 0 & 2^+ & 2^+ \end{vmatrix} \qquad
\begin{vmatrix} 2^+ & 2^+ & 0 \\ 2^+ & 0 & 2^+ \end{vmatrix} \qquad
\begin{vmatrix} 2^+ & 2^+ & 0 \\ 0 & 0 & 2^+ \end{vmatrix}
$$

We will have at least a little to say about each of these forms.

3.5 The form $\begin{vmatrix} 3 & 3 & 1 \\ 3 & 1 & 3 \end{vmatrix}$

There are cuts for $\begin{vmatrix} 3^+ & 5^+ & 1 \\ 3^+ & 1 & 5^+ \end{vmatrix}$. Apparently there is no cut of $\begin{vmatrix} 3 & 3 & 1 \\ 3 & 1 & 3 \end{vmatrix}$, but there is a lift. The cases $\begin{vmatrix} 5^+ & 3 & 1 \\ 3^+ & 1 & 3^+ \end{vmatrix}$ and $\begin{vmatrix} 3^+ & 3 & 1 \\ 3^+ & 1 & 5^+ \end{vmatrix}$ have been vexing and remain open.

Theorem 3.4 $\begin{bmatrix} 3 & 3 & 1 \\ 3 & 1 & 3 \end{bmatrix}$

Proof Define new symbols $\mathsf{C}, \mathbf{C}, \overline{\mathsf{C}}, \overline{\mathbf{C}}$, mapping to $C, \overline{C}$, and $\gamma, \overline{\gamma}$ mapping to $\gamma, \overline{\gamma}$; define $\mathcal{A}^+$, $\mathcal{L}^+$, and $\mathcal{R}^+$ as indicated below— the letters of $\mathcal{A}^+$ are on the right; the language follows the same method as $\mathcal{L}_T$ (adjacent symbols must match) and the rules are as given. It is not hard to see that these form a lift for $\mathcal{A}_\mathcal{V}, \mathcal{L}_\mathcal{V}, \mathcal{R}_\mathcal{V}$.

$$\begin{array}{rl@{\qquad}rl}
\mathsf{C}\overline{\mathsf{C}} \to \mathsf{C}\overline{\mathsf{C}}\mathsf{C}\gamma\mathbf{C} & \beta\overline{\beta}\beta\gamma\overline{\gamma}\gamma\alpha & \mathbf{C}\overline{\mathbf{C}} \to \overline{\mathbf{C}}\mathbf{C}\overline{\mathbf{C}}\overline{\gamma}\overline{\mathsf{C}} & \beta\overline{\beta}\overline{\alpha\gamma}\gamma\overline{\gamma}\overline{\beta} \\
\overline{\mathsf{C}}\mathsf{C} \to \mathbf{C}\overline{\mathbf{C}}\mathbf{C}\gamma\mathsf{C} & \overline{\alpha}\alpha\beta\gamma\overline{\gamma}\gamma\alpha & \overline{\mathbf{C}}\mathbf{C} \to \overline{\mathsf{C}}\mathsf{C}\overline{\mathsf{C}}\overline{\gamma}\overline{\mathbf{C}} & \overline{\alpha}\alpha\overline{\alpha\gamma}\gamma\overline{\gamma}\overline{\beta} \\
\mathsf{C}\gamma\mathbf{C} \to \mathsf{C}\overline{\mathsf{C}}\overline{\gamma}\overline{\mathbf{C}} & \beta\gamma\alpha\overline{\alpha\gamma}\gamma\alpha & \mathbf{C}\gamma\mathsf{C} \to \overline{\mathbf{C}}\mathbf{C}\gamma\mathsf{C} & \beta\gamma\alpha\beta\gamma\overline{\gamma}\overline{\beta} \\
\overline{\mathsf{C}}\overline{\gamma}\overline{\mathbf{C}} \to \mathbf{C}\overline{\mathbf{C}}\overline{\gamma}\overline{\mathsf{C}} & \overline{\alpha\gamma}\overline{\beta}\overline{\alpha\gamma}\gamma\alpha & \overline{\mathbf{C}}\overline{\gamma}\overline{\mathsf{C}} \to \overline{\mathsf{C}}\mathsf{C}\gamma\mathbf{C} & \overline{\alpha\gamma}\overline{\beta}\beta\gamma\overline{\gamma}\overline{\beta}
\end{array}$$

□

Theorem 3.5 $\begin{bmatrix} 3^+ & 5^+ & 1 \\ 3^+ & 1 & 5^+ \end{bmatrix}$

Proof We take the following cut for $\begin{bmatrix} 2p+3 & 2q+5 & 1 \\ 2r+3 & 1 & 2s+5 \end{bmatrix}$, $p, q, r, s \geq 0$. Note that this is in fact a cut by applying Lemma 3.2 using the function $f(A, B, C, \overline{C}, \overline{B}, \overline{A}) = (C, B, B, \overline{C}, \overline{C}, \overline{B})$. The rules are:

$$\begin{array}{lll}
[BA] \to B\overline{B}^\rho BA\overline{A}\overline{\alpha}C & [B\overline{B}] \to B\overline{B}^\rho B\overline{B}B\beta\overline{C} & [CB] \to B\overline{B}^\rho BA\overline{A}\beta B \\
[C\overline{C}] \to B\overline{B}^\rho\overline{CA}A\overline{\gamma}\overline{C} & [\overline{A}A] \to \overline{B}^\rho\overline{CA}A\overline{A}\overline{\alpha}C & [\overline{AB}] \to \overline{B}^\rho\overline{CAB}B\beta\overline{C} \\
[B\beta\overline{C}] \to B\overline{B}^\rho BA\overline{\gamma}\overline{C} & [\overline{A}\beta B] \to \overline{B}^\rho BA\overline{A}\beta B & [\overline{A}\overline{\alpha}C] \to \overline{B}^\rho B\overline{BC}\overline{\gamma}B
\end{array}$$

$$\begin{array}{lll}
[AC] \to C^\rho BAC\overline{C}\overline{\gamma}B & [A\overline{A}] \to C^\rho BA\overline{A}A\alpha\overline{B} & [\overline{B}B] \to \overline{C}C^\rho BA\overline{A}\beta B \\
[\overline{BC}] \to \overline{C}C^\rho\overline{CA}A\overline{\gamma}\overline{C} & [\overline{C}C] \to \overline{C}C^\rho\overline{C}C\overline{C}\overline{\gamma}B & [\overline{CA}] \to \overline{C}C^\rho\overline{CA}A\alpha\overline{B} \\
[A\alpha\overline{B}] \to C^\rho\overline{C}CB\beta\overline{C} & [A\overline{\gamma}\overline{C}] \to C^\rho\overline{CA}A\overline{\gamma}\overline{C} & [\overline{C}\overline{\gamma}B] \to \overline{C}C^\rho\overline{CA}\beta B
\end{array}$$

where $\overline{B}^{rho} = (\overline{B}B)^q(A\overline{A})^p\overline{B}$ and $C^\rho = (C\overline{C})^s(\overline{A}A)^rC$. □

3.6 The forms $\begin{vmatrix} 3^+ & 3^+ & 1 \\ 2^+ & 0 & 2^+ \end{vmatrix}$ and $\begin{vmatrix} 2^+ & 2^+ & 0 \\ 2^+ & 0 & 2^+ \end{vmatrix}$

In fact there are some lovely geometric methods for more completely understanding these two forms. In the meanwhile, however, through the same methods we have been applying so far, we prove:

Theorem 3.6

$$\begin{bmatrix} 5^+ & 5^+ & 1 \\ 4^+ & 0 & 6^+ \end{bmatrix} \qquad \begin{bmatrix} 5^+ & 7^+ & 1 \\ 6^+ & 0 & 4^+ \end{bmatrix} \qquad \begin{bmatrix} 4^+ & 8^+ & 0 \\ 4^+ & 0 & 8^+ \end{bmatrix} \qquad \begin{bmatrix} 6^+ & 6^+ & 0 \\ 4^+ & 0 & 8^+ \end{bmatrix}$$

Proof For the following, let $p, q, r, s \geq 0$ and set $\overline{B}^\rho = (\overline{B}B)^q(A\overline{A})^p\overline{B}$, $C^\rho = (C\overline{C})^s(\overline{A}A)^rC$, $A^\rho = (A\overline{A})^p(\overline{B}B)^qA$, and $\overline{A}^\rho = (\overline{A}A)^r(C\overline{C})^sA$.

For $\begin{bmatrix} 4+2p & 8+2q & 0 \\ 4+2r & 0 & 8+2s \end{bmatrix}$, $p, q, r, s \geq 0$, the following rules form a cut, applying Lemma 3.2 with $f(A, B, C, \overline{C}, \overline{B}, \overline{A}) = (C, \overline{B}, B, C, \overline{C}, \overline{B})$:

$$\begin{array}{lll}
[BA] \to \overline{B}^\rho B\overline{B}B\overline{B}BA\overline{A}\overline{\alpha}C & [B\overline{B}] \to \overline{B}^\rho B\overline{B}B\overline{B}B\overline{B}B\beta\overline{C} & [C\overline{C}] \to B\overline{B}^\rho BA\overline{A}A\overline{AB}\gamma C \\
[\overline{A}A] \to \overline{B}^\rho B\overline{B}BA\overline{A}A\overline{A}\overline{\alpha}C & [\overline{AB}] \to \overline{B}^\rho B\overline{B}B\overline{B}BA\overline{A}\beta\overline{C} & [B\beta\overline{C}] \to \overline{B}^\rho BA\overline{AB}B\overline{B}\gamma C \\
[C\overline{\beta B}] \to B\overline{B}^\rho B\overline{B}BA\overline{A}\beta\overline{C} & [\overline{A}\beta\overline{C}] \to \overline{B}^\rho BA\overline{A}A\overline{AB}\gamma C & [\overline{A}\overline{\alpha}C] \to \overline{B}^\rho B\overline{B}B\overline{B}BA\overline{\gamma}B \\
[AC] \to C^\rho\overline{C}C\overline{C}C\overline{CA}A\overline{\gamma}B & [A\overline{A}] \to C^\rho\overline{C}C\overline{CA}A\overline{A}A\alpha\overline{B} & [\overline{B}B] \to \overline{C}C^\rho\overline{CA}A\overline{A}AC\overline{\beta B} \\
[\overline{C}C] \to C^\rho\overline{C}C\overline{C}C\overline{C}C\overline{C}\overline{\gamma}B & [\overline{CA}] \to C^\rho\overline{C}C\overline{C}C\overline{CA}A\alpha\overline{B} & [A\alpha\overline{B}] \to C^\rho\overline{C}C\overline{C}C\overline{CA}\beta\overline{C} \\
[A\overline{\gamma}B] \to C^\rho\overline{CA}A\overline{A}AC\overline{\beta B} & [\overline{B}\gamma C] \to \overline{C}C^\rho\overline{C}C\overline{CA}A\overline{\gamma}B & [\overline{C}\overline{\gamma}B] \to C^\rho\overline{CA}AC\overline{C}C\overline{\beta B}
\end{array}$$

For $\begin{bmatrix} 6+2p & 6+2q & 0 \\ 4+2r & 0 & 8+2s \end{bmatrix}$, $p, q, r, s \geq 0$, the following rules form a cut, applying Lemma 3.2 with $f(A, B, C, \overline{C}, \overline{B}, \overline{A}) = (C, A, B, \overline{A}, \overline{C}, \overline{B})$:

$$\begin{array}{lll}
[BA] \to A^\rho \overline{AB} B \overline{B} B A \overline{A} \overline{\alpha} C & [B\overline{B}] \to A^\rho \overline{AB} B \overline{B} B \overline{B} B \beta \overline{C} & [C\overline{C}] \to B A^\rho \overline{A} A \overline{A} A \overline{AB} \gamma \overline{A} \\
[\overline{A}A] \to \overline{B} B A^\rho \overline{A} A \overline{A} A \overline{A} \overline{\alpha} C & [\overline{AB}] \to \overline{B} B A^\rho \overline{A} A \overline{AB} B \beta \overline{C} & [\overline{C}C] \to \overline{AB}^\rho B \overline{B} B \overline{B} B A \overline{\gamma} B \\
[B\beta\overline{C}] \to A^\rho \overline{A} A \overline{AB} B \overline{B} \gamma \overline{A} & [C\overline{\beta}A] \to B A^\rho \overline{A} A \overline{A} A \overline{A} \overline{\alpha} C & [\overline{A}\overline{\alpha}C] \to \overline{B}^\rho B \overline{B} B A \overline{A} A \overline{\gamma} B \\
[\overline{C}\alpha\overline{B}] \to \overline{AB}^\rho B \overline{B} B \overline{B} B \beta \overline{C} & [AC] \to C^\rho \overline{C} C \overline{C} C \overline{CA} A \overline{\gamma} B & [A\overline{A}] \to C^\rho \overline{CA} A \overline{A} A C \overline{C} \alpha \overline{B} \\
[\overline{B}B] \to \overline{CA} A \overline{A} A C^\rho \overline{C} C \overline{\beta} A & [\overline{CA}] \to \overline{A} A C^\rho \overline{C} C \overline{C} C \overline{C} \alpha \overline{B} & [A\overline{\gamma}B] \to C^\rho \overline{CA} A \overline{A} A C \overline{\beta} A \\
[\overline{B}\gamma\overline{A}] \to \overline{CA} A \overline{A} A C^\rho \overline{C} \alpha \overline{B} & &
\end{array}$$

For $\begin{bmatrix} 4+2p & 6+2q & 0 \\ 5+2r & 1 & 5+2s \end{bmatrix}$, $p, q, r, s \geq 0$, the following rules form a cut, applying Lemma 3.2 with $f(A, B, C, \overline{C}, \overline{B}, \overline{A}) = (C, A, \overline{B}, \overline{A}, B, \overline{B})$:

$$\begin{array}{lll}
[BA] \to A \overline{AB}^\rho B \overline{B} B \overline{\alpha} C & [C\overline{C}] \to \overline{A} A \overline{AB}^\rho B \overline{B} \gamma \overline{A} & [\overline{A}A] \to \overline{B}^\rho B A \overline{A} A \overline{A} \overline{\alpha} C \\
[\overline{AB}] \to \overline{B}^\rho B A \overline{AB} B \beta B & [\overline{C}C] \to \overline{AB}^\rho B \overline{B} B \overline{B} \gamma \overline{A} & [C\overline{\beta}A] \to \overline{A} A \overline{AB}^\rho B \overline{\alpha} C \\
[C\overline{\beta}A] \to \overline{AB}^\rho B A \overline{A} \overline{\alpha} C & [\overline{A}\overline{\alpha}C] \to \overline{B}^\rho B A \overline{AB} \gamma \overline{A} & [AC] \to C^\rho \overline{CA} A \overline{A} A C \gamma \overline{A} \\
[A\overline{A}] \to C^\rho B A \overline{A} A \overline{A} A \alpha \overline{B} & [B\overline{B}] \to A C^\rho \overline{C} C \overline{C} C B \beta B & [CB] \to \overline{A} A \overline{A} A C^\rho \overline{C} C \overline{\beta} A \\
[\overline{B}B] \to B A \overline{A} A \overline{A} A C^\rho \overline{\beta} A & [\overline{C}C] \to \overline{A} A C^\rho \overline{C} C \overline{C} C \gamma \overline{A} & [\overline{CA}] \to \overline{A} A C^\rho \overline{C} C B A \alpha \overline{B} \\
[A\alpha\overline{B}] \to C^\rho \overline{CA} A C B \beta B & [B\beta B] \to A \overline{A} A C^\rho \overline{C} C \overline{\beta} A & [B\overline{\alpha}C] \to A C^\rho \overline{C} C \overline{C} C \gamma \overline{A} \\
[C\gamma\overline{A}] \to \overline{A} A C^\rho \overline{CA} A \alpha \overline{B} & [\overline{B}\gamma\overline{A}] \to B A \overline{A}^\rho A \overline{A} A \alpha \overline{B} &
\end{array}$$

For $\begin{bmatrix} 6+2p & 4+2q & 0 \\ 5+2r & 1 & 7+2s \end{bmatrix}$, $p, q, r, s \geq 0$, the following rules form a cut, applying Lemma 3.2 with $f(A, B, C, \overline{C}, \overline{B}, \overline{A}) = (C, A, \overline{B}, \overline{A}, B, A)$:

$$\begin{array}{lll}
[BA] \to A^\rho \overline{AB} B A \overline{A} \overline{\alpha} C & & \\
[B\overline{B}] \to A^\rho \overline{AB} B \overline{B} B \beta B & [C\overline{C}] \to \overline{A} A^\rho \overline{A} A \overline{AB} \gamma \overline{A} & [\overline{A}A] \to A^\rho \overline{A} A \overline{A} A \overline{A} \overline{\alpha} C \\
[\overline{AB}] \to A^\rho \overline{A} A \overline{AB} B \beta B & [\overline{C}C] \to \overline{A} A^\rho \overline{AB} B \overline{B} \gamma \overline{A} & [C\overline{\beta}A] \to \overline{A} A^\rho \overline{A} A \overline{A} \overline{\alpha} C \\
[\overline{A}\overline{\alpha}C] \to A^\rho \overline{A} A \overline{AB} \gamma \overline{A} & [\overline{C}\alpha A] \to \overline{AB} B A^\rho \overline{A} \overline{\alpha} C & [AC] \to C^\rho \overline{C} C \overline{CA} A \overline{A} A C \gamma \overline{A} \\
[A\overline{A}] \to C^\rho B A \overline{A} A \overline{A} A C \overline{C} \alpha A & [B\overline{B}] \to A C^\rho \overline{C} C \overline{C} C \overline{C} C B \beta B & [CB] \to \overline{A} A \overline{A} A C^\rho \overline{C} C \overline{C} C \overline{\beta} A \\
[\overline{B}B] \to B A \overline{A} A \overline{A} A C^\rho \overline{C} C \overline{\beta} A & [\overline{C}C] \to \overline{A} A C \overline{C} C^\rho \overline{C} C \overline{C} C \gamma \overline{A} & [\overline{CA}] \to \overline{A} A C^\rho B A C \overline{C} C \overline{C} \alpha A \\
[B\beta B] \to A \overline{A} A C^\rho \overline{C} C \overline{C} C \overline{\beta} A & [C\gamma\overline{A}] \to \overline{A} A \overline{A} A C^\rho \overline{C} C \overline{C} \alpha A & [\overline{B}\gamma\overline{A}] \to B A \overline{A} A \overline{A} A C^\rho \overline{C} \alpha A
\end{array}$$

$\square$

3.7 More interesting examples

We now turn to more interesting examples. We first will consider a regular production system describing tilings by p-gons meeting q-to-a-vertex, $p, q \geq 4$; next we will lift this to a system describing the case $\begin{bmatrix} 4^+ & 4^+ & 0 \\ 0 & 0 & 4^+ \end{bmatrix}$. Here the lifting corresponds to various ways of subdividing the polygons into triangles.

We next wish apply the topological method described in Lemma 2.1 to prove $\begin{bmatrix} 5^+ & 5^+ & 1 \\ 0 & 0 & 4^+ \end{bmatrix}$ and $\begin{bmatrix} 5^+ & 5^+ & 1 \\ 1 & 1 & 5^+ \end{bmatrix}$. However this requires selecting certain edges in the tilings by $\begin{bmatrix} 4^+ & 4^+ & 0 \\ 0 & 0 & 4^+ \end{bmatrix}$. We make this selection by lifting once again.

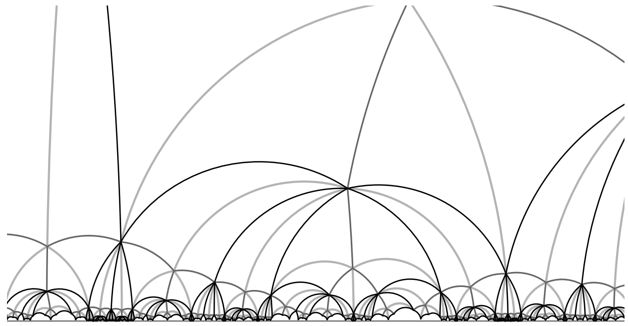
It is worth noting that we can construct lifts of our original productions on triangles that will precisely capture all the structure we use here. Indeed, in principle, this is not intrinsically difficult, but would be exceedingly tedious.

Following a few more minor observations, we will have completely sorted out the forms $\begin{vmatrix} 2^+ & 2^+ & 0 \\ 0 & 0 & 2^+ \end{vmatrix}$ and $\begin{vmatrix} 3^+ & 3^+ & 1 \\ 1 & 1 & 3^+ \end{vmatrix}$, leaving a little bit of $\begin{vmatrix} 3^+ & 3^+ & 1 \\ 0 & 0 & 2^+ \end{vmatrix}$.

3.7.1 $\begin{vmatrix} 2^+ & 2^+ & 0 \\ 0 & 0 & 2^+ \end{vmatrix}$

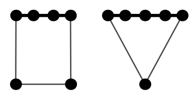
We begin with a regular production system that models regular p-gons meeting q-to-a-vertex, with $p, q \geq 4$. We will then take a certain lift of this system that is useful for:

Theorem 3.7 $\begin{bmatrix} 4^+ & 4^+ & 0 \\ 0 & 0 & 4^+ \end{bmatrix}$



Note that with Lemma 1.1, this completely settles the form $\left|\begin{smallmatrix} 2^+ & 2^+ & 0 \\ 0 & 0 & 2^+ \end{smallmatrix}\right|$.

Proof We will prove $\left[\begin{smallmatrix} 2i & 2j & 0 \\ 0 & 0 & 2k \end{smallmatrix}\right]$, $i, j, k \geq 2$ by constructing a subdivision of the tiling by $(2k)$-gons meeting $(i+j)$-to-a-vertex. Let $p = 2k$, $q = i + j$ and consider the alphabet $\square, \triangledown$ representing the configurations



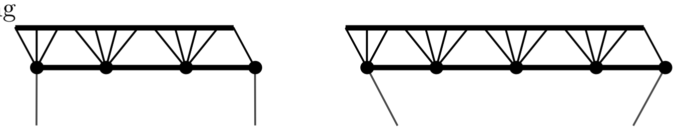
shown here with $p = 6$.

Take the free production system on $\square, \triangledown$ defined by

$$\square \mapsto \triangledown^{q-4}\square(\triangledown^{q-3}\square)^{p-4}$$

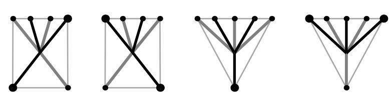
$$\triangledown \mapsto \triangledown^{q-4}\square(\triangledown^{q-3}\square)^{p-3}$$

encoding the layering



shown here with $p = q = 6$. Clearly orbits in the system correspond to tilings by abstract p-gons meeting q-to-a-vertex.

We now take the following lift, with alphabet $\square_{/}$, $\square_{\backslash}$, $\triangledown_{\bot}$, $\triangledown_{\top}$ and obvious map onto $\square, \triangledown$, representing the configurations



shown here with $i = j = 3$.

We take the productions, on the free language in our alphabet:

$$\triangledown_{\top} \mapsto \triangledown_{\top}^{j-2}\ \triangledown_{\bot}^{i-2}\ \square_{\backslash}\ (\ \triangledown_{\top}^{j-1}\ \triangledown_{\bot}^{i-2}\ \square_{/}\ \triangledown_{\top}^{j-2}\ \triangledown_{\bot}^{i-1}\ \square_{\backslash})^{k-2}\ \triangledown_{\top}^{j-1}\ \triangledown_{\bot}^{i-2}\ \square_{/}$$

$$\triangledown_{\bot} \mapsto \triangledown_{\top}^{j-2}\ \triangledown_{\bot}^{i-2}\ \square_{/}\ (\ \triangledown_{\top}^{j-2}\ \triangledown_{\bot}^{i-1}\ \square_{\backslash}\ \triangledown_{\top}^{j-1}\ \triangledown_{\bot}^{i-2}\ \square_{/})^{k-2}\ \triangledown_{\top}^{j-2}\ \triangledown_{\bot}^{i-1}\ \square_{\backslash}$$

$$\square_{/} \mapsto \triangledown_{\top}^{j-2}\ \triangledown_{\bot}^{i-2}\ \square_{/}\ (\ \triangledown_{\top}^{j-2}\ \triangledown_{\bot}^{i-1}\ \square_{\backslash}\ \triangledown_{\top}^{j-1}\ \triangledown_{\bot}^{i-2}\ \square_{/})^{k-2}$$

$$\square_{\backslash} \mapsto \triangledown_{\top}^{j-2}\ \triangledown_{\bot}^{i-2}\ \square_{\backslash}\ (\ \triangledown_{\top}^{j-1}\ \triangledown_{\bot}^{i-2}\ \square_{/}\ \triangledown_{\top}^{j-2}\ \triangledown_{\bot}^{i-1}\ \square_{\backslash})^{k-2}$$

By inducting on superwords one may easily verify that at every vertex in any complex represented by an orbit in this system, we have a vertex arrangement in $\mathcal{V}\,(2p\ 2q\ 0)$ (at the vertices of our original $(2r)$-gons) or $\mathcal{V}\,(0\ 0\ p+q)$ (at the centers of our original $(2r)$-gons). There exist orbits in the resulting system and each orbit corresponds to a $\mathcal{V}$-complex, $\mathcal{V} = \mathcal{V}\left(\begin{smallmatrix} 2i & 2j & 0 \\ 0 & 0 & 2k \end{smallmatrix}\right)$. $\square$

3.8 $\begin{vmatrix} 3^+ & 3^+ & 1 \\ 0 & 0 & 2^+ \end{vmatrix}$

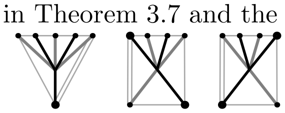
We pause for the following lemma; the proof is an elementary check:.

Lemma 3.8 *There are no 2-nions by vertices in* $\mathcal{V}\begin{pmatrix} 5^+ & 3 & 1 \\ 0 & 0 & 2^+ \end{pmatrix}$.

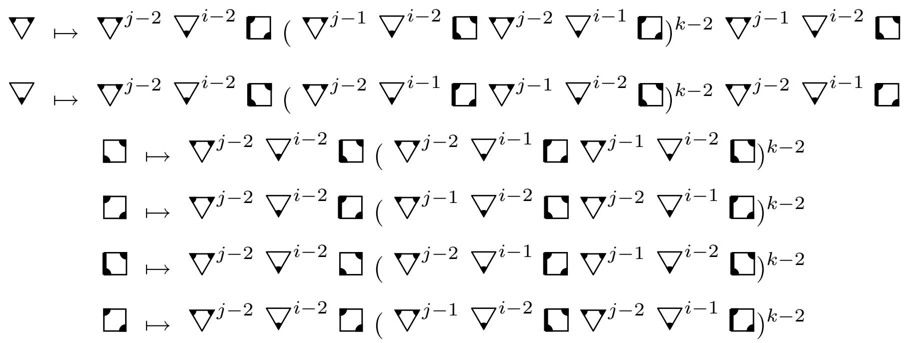
With Lemma 1.1, Corollary 2.2, and Lemma 3.8, the following theorem completely sorts out the form $\begin{vmatrix} 3^+ & 3^+ & 1 \\ 0 & 0 & 2^+ \end{vmatrix}$.

Theorem 3.9 $\begin{bmatrix} 5^+ & 5^+ & 1 \\ 0 & 0 & 4^+ \end{bmatrix}$.

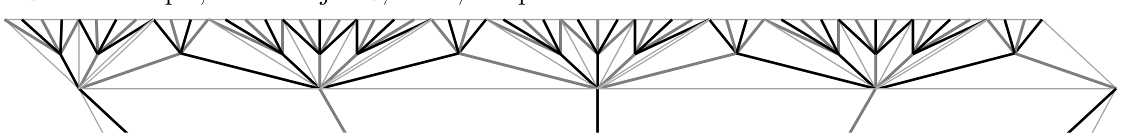
Proof We wish to apply the trick in Lemma 2.1, starting with tilings of the form $\begin{bmatrix} 4^+ & 4^+ & 0 \\ 0 & 0 & 4^+ \end{bmatrix}$; but we must specify the sets of edges $\mathcal{E}$. To show $\begin{bmatrix} 2i+1 & 2j+1 & 1 \\ 0 & 0 & 2k \end{bmatrix}$, $i, j, k \geq 2$, we lift the system of Theorem 3.7 as follows: we take the alphabet ▽, ▽, □, □, □, □, and □; the first four letters represent the same configurations as in Theorem 3.7 and the last two represent:



respectively. The "hollow" lines represent the special edges in $\mathcal{E}$. We take the productions:



So for example, with $i = j = 3, k = 1$, the production for ▽ encodes



As before by inducting on superwords one may easily verify that at every vertex in any complex represented by an orbit in this system, we have a vertex arrangement in $\mathcal{V}((2i+1)\ (2j+1)\ 1)$ or $\mathcal{V}(1\ 1\ 2k)$; and that every vertex in $\mathcal{V}((2i+1)\ (2j+1)\ 1)$ meets exactly one edge in $\mathcal{E}$. Orbits in the system correspond to $\mathcal{V}$-complices with exactly one edge meeting each vertex in $\mathcal{V}((2i+1)\ (2j+1)\ 1)$, and we may apply Lemma 2.1 to obtain $\begin{bmatrix} 2i+1 & 2j+1 & 1 \\ 0 & 0 & 2k \end{bmatrix}$. □

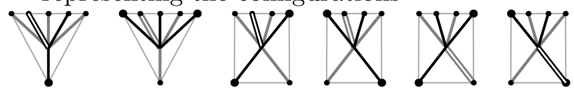
3.8.1 $\begin{vmatrix} 3^+ & 3^+ & 1 \\ 1 & 1 & 3^+ \end{vmatrix}$

The cases $\begin{vmatrix} p & q & 1 \\ 1 & 1 & 3 \end{vmatrix}$, p, q odd, and $\begin{vmatrix} 5^+ & 3 & 1 \\ 1 & 1 & 5^+ \end{vmatrix}$ are open, for the moment. Other than those, with Corollary 2.2 and Theorem 2.4, the following sorts out the case $\begin{vmatrix} 3^+ & 3^+ & 1 \\ 1 & 1 & 3^+ \end{vmatrix}$:

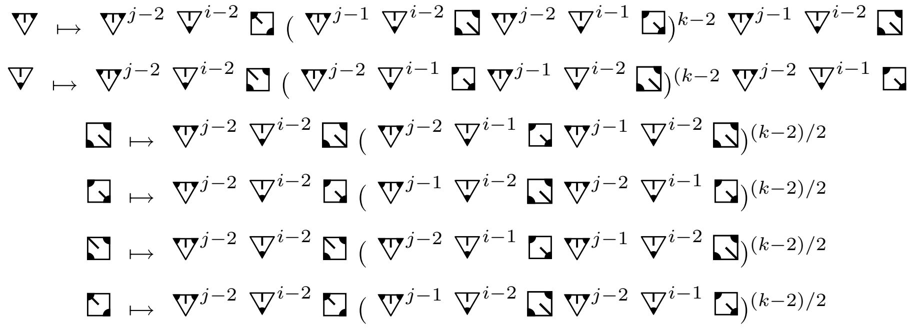
Theorem 3.10 $\begin{bmatrix} 5^+ & 5^+ & 1 \\ 1 & 1 & 5^+ \end{bmatrix}$

Proof As in Theorem 3.9, we lift the production of Theorem 3.7 in order to mark a set of edges $\mathcal{E}$ so we may apply Lemma 2.1. We show $\begin{bmatrix} 2i+1 & 2j+1 & 1 \\ 1 & 1 & 2k+1 \end{bmatrix}$, $i, j, k \geq 2$. We use the following alphabet:

, and  representing the configurations



We use the following productions:



Once again, it is not hard to show that in any complex corresponding to orbits in this system, all vertices are in $\mathcal{V}\begin{pmatrix} 2i+1 & 2j+1 & 1 \\ 1 & 1 & 2k+1 \end{pmatrix}$ and are incident to exactly one edge in $\mathcal{E}$. Hence we may apply Lemma 2.1. $\square$

3.9 Remaining triangles

We have not discussed production systems for vertex arrangements of fewer than six triangles; though we did examine many tilings with these arrangements in Section 2 and Section 3.2 of [16]. With some extra care, it is however, perfectly possible to define production systems that capture such vertex arrangements.

We have not explicitly discussed tilings by isoceles or equilateral triangles. These are actually much simpler to describe than scalene triangles, and we leave this as an extended exercise.

We have not systematically examined the remaining 0-cells in $\cap\Pi$. It seems this should wait until one has exhausted the families of 1-cells in $\cap\Pi$. These don't seem impossibly difficult, merely time-consuming.

Though the tools permit some analysis, we have not examined the *space* of tilings admitted by a given triangle, but merely tried to answer which triangles admit tilings. These spaces (i.e. the sets of orbits) seem quite interesting, from a number of vantage points.

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