Open Questions in Tiling

Chaim Goodman-Strauss

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Abstract

In the following survey, we consider connections between several open questions regarding tilings in general settings. Along the way, we support a careful revision of the definition of aperiodicity, and pose several new conjectures. We also give several new examples.

Over the years a number of interesting questions have been asked about the combinatorial complexity of tilings in the plane or other spaces. Here we consider several of these questions, and their connections, in a very general setting.

Our most general question is simply: "How complex can the behaviour of a given protoset be?"

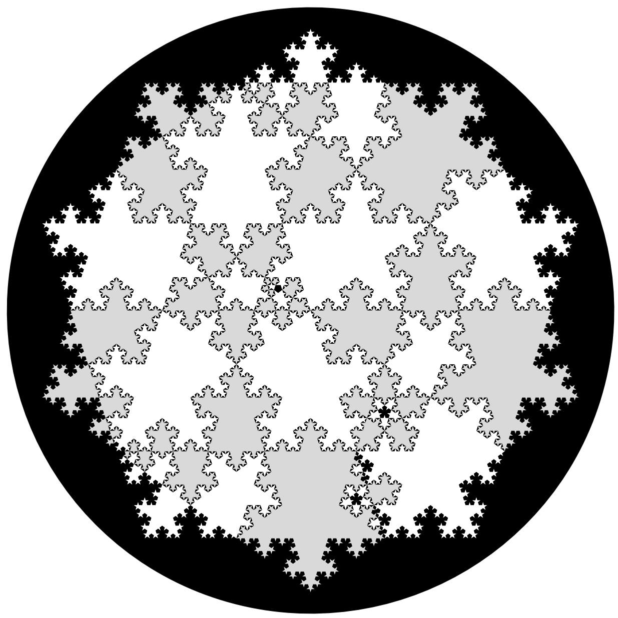

Figure 1: A monohedral tiling

A protoset may behave in many ways: First, a protoset might admit no tilings whatsoever. In this case, as long as the setting isn't too pathological, there is some upper bound on the size of configurations that this protoset can form (for if the protoset admits arbitratrily large configurations, one can produce a tiling of the entire space; cf. Theorem 3.8.1 in [16]). One measure of this bound is the Heesch number of the protoset, defined below.

Or if a protoset does admit tilings, it might admit "strongly periodic" tilings— that is tilings with a symmetry that has a compact fundamental domain; or it might admit only "weakly periodic" tilings— tilings with a infinite cyclic symmetry, but not necessarily a compact fundamental domain. (In the Euclidean plane, ruling out certain pathologies, weak and strong periodicity coincide (see Theorem 3.7.1 in [16]), though not in higher dimensions or in hyperbolic space.) If the protoset admits strongly periodic tilings then there is some lower bound on the number of orbits of the prototiles in such a tiling. One measure of this is the isohedral number of the protoset defined below.

We have a third outcome: a protoset admits tilings but no periodic tilings whatsoever. In this case, we can ask if the protoset is "aperiodic", a most remarkable property. We can go further, and even ask if the protoset is "non-recursive", that is, does the protoset admit tilings but only tilings that cannot be produced by any algorithm (local or otherwise) [18, 31].

In any case we can ask, in a given fixed setting (eg. $\mathbb{H}^2$ with polygonal tiles), is there a bound on the Heesch or isohedral numbers of protosets? Is there an aperiodic protoset? Is there a non-recursive protoset?

That is, in a given setting, *how complex can the simplest allowed behaviour of a protoset be?*

We discuss these questions, their tight interconnections, and their history, and give a few conjectures on which we are actively working, and on which we invite the reader to work as well! We are grateful to Jarkko Kari, Michael Hartley, Serge Tabachnikov and Peter Schmitt for illuminating conversation, and to Asia Weiss and Javier Bracho for the opportunity to give a series of lectures at the Instituto de Matemáticas (UNAM) from which this survey grew.

1 First Definitions

Though tilings may considered in much wider settings, for now we restrict ourselves to the following: Let X be a non-compact metric n-manifold; let $\mathcal{G}$ consist of some group of metric-preserving automorphisms of X.

Note as general as this seems, the perfectly sensible monohedral tiling of figure 1 (briefly discussed in Section 3.1.5) lies outside this setting— there we allow scalings of the single prototile.

In [15] we will give a more comprehensive discussion of tiles, markings, matching rules and so forth, placing these ideas on firm set theoretic footing. For now, a **tile** is some compact set of points that is the closure of its interior. We might specify some additional constraints, such as requiring the tiles to be polygonal, etc. We denote a set of some constraints R. Given a finite set T of tiles (often called a **protoset** of **prototiles**) a **configuration of tiles in** T is a collection $\tau = \{gA\}$ of tiles in $\mathcal{G}T$ such that for all $gA, hB \in \tau$, gA and hB have disjoint interiors. The **support** of τ is simply the union of the tiles in τ. A **tiling** is a configuration with support X; a **species** Σ is any collection of tilings. The species $\Sigma(T)$ is simply the species of all possible tilings of X by prototiles in T.

Now one can place additional restrictions on tilings in a given species. These can take many forms. We might require any number of global conditions (that the tilings arise from a substitution system, or are quasiperiodic, etc.), or local conditions (that the tilings satisfy some local "matching rules"). Matching rules in particular seem to take many different forms: we can for example, endow our tiles with markings and require that adjacent tiles have compatible marks; or we can require that the tilings can be covered by some suitable, finite atlas of permitted, bounded configurations; or we can require no more than that the tiles fit together. In [15] all this variety is subsumed into a single well-defined structure— the "mingle"; but for now, let us keep the discussion of matching rules informal. For any given set of local restrictions $\mathcal{M}$ on $\Sigma(T)$, let $\Sigma(T, \mathcal{M})$ consist of all tilings in $\Sigma(T)$ satisfying $\mathcal{M}$. Restrictions on the kinds of matching rules we work with will be folded into R, restrictions on the protoset.

Thus, a given setting is specified by X, $\mathcal{G}$, and R. For example, we might consider tilings of $\mathbb{H}^2$, by polygonal tiles meeting vertex-to-vertex; or tilings of $\mathbb{E}^2$, by translations of square tiles with colored edges, requiring the colors of coinciding edges to match. Or we might consider tilings of $\mathbb{E}^3$ by a single polyhedral prototile ("monohedral" tilings). And so forth.

2 The Completion and Domino Problems

In a fixed, specific setting $(X, \mathcal{G}, R)$, we can ask the following:

Question 2.1 *Is there an algorithm that, upon being given a set of prototiles T and a configuration C of these tiles, decides whether or not there is a tiling $\tau \in \Sigma(T)$ with $C \subset \tau$? That is, is the "Completion Problem" decidable?*

Question 2.2 *Is there an algorithm that, upon being given a set of prototiles T decides whether or not $\Sigma(T) = \emptyset$? That is, is the "Domino Problem" decidable?*

Initially, the above questions were framed in the Euclidean plane. In 1961 Wang showed that the Completion Problem, for square tiles with colored edges moved by translations only, in $\mathbb{E}^2$ ("Wang tiles"), is undecidable by constructing, for any Turing machine, a set of tiles T so that a certain "seed" configuration could be completed to a tiling in $\Sigma(T)$ if and only if the machine fails to halt [48].

At left in figure 2 is a very rough schematic of Wang's construction as interpreted in [41]. Each row of the tiling corresponds to a time in the run of the Turing machine— the symbols on the machine's tape, and the position and state of the machine's head. The tiles are designed to reflect the possible transitions from one row to the next. We must assume a particular starting configuration is used to get the machine rolling. The tiling can be completed if and only if the corresponding machine never halts.

Since the Halting Problem is undecidable, so too is the Completion Problem. Wang asked Question 2.2, and conjectured the answer was positive. In particular, he could not see how one could construct a set of tiles so that the use of the seed configuration he required could be guaranteed (note that the Domino problem *is* decidable for the tiles in the above construction).

In 1964, Berger proved that the Domino Problem was in fact undecidable [4] for Wang tiles in $\mathbb{E}^2$. In particular, then, Berger was able to construct, for any given Turing machine, a set T of tiles such that $\Sigma(T) \neq \emptyset$ if and only if the corresponding machine fails to halt. He did this by first building a "hierarchical framework"— a hierarchy of larger and larger domains forced to appear by the structure of the tiles in the protoset — on which to hang Wang's construction. A lovely reworking of both proofs appears in [41]. By and large, if we regard other classes of protosets in $\mathbb{E}^2$ or higher dimensonal Euclidean spaces, these results can be extended.

However, the answer to either of these questions is unknown if we ask that T contain only a single prototile. This conjecture has surely occured to others:

Conjecture 2.3 *In the Euclidean plane, among polygonal monohedral tilings, the Completion problem and the Domino problem are decidable.*

Note that this is immediately true if it turns out there are only finitely many combinatorially distinct monotiles.

In 1977, R.M. Robinson showed that the Completion Problem is undecidable in $\mathbb{H}^2$, for marked, right-angled hexagonal tiles [42]. His result really is not surprising, as it models Wang's original result on square tiles within a tiling of $\mathbb{H}^2$ such as that of figure 3. The basic idea is that, if one can guarantee the placement of a seed tile, then one can encode the Euclidean square lattice within this tiling of $\mathbb{H}^2$.

On the right in figure 2 we suggest how this construction can be carried over to $\mathbb{H}^2$: the white squares model the registers of the Turing machine; but note that Robinson requires the use of increasingly large regions, shaded gray, of tiles that imitate the edges of the original square lattice in $\mathbb{E}^2$; without the use of the seed tile, one might end up with a (weakly aperiodic) tiling of only these "edge-tiles".

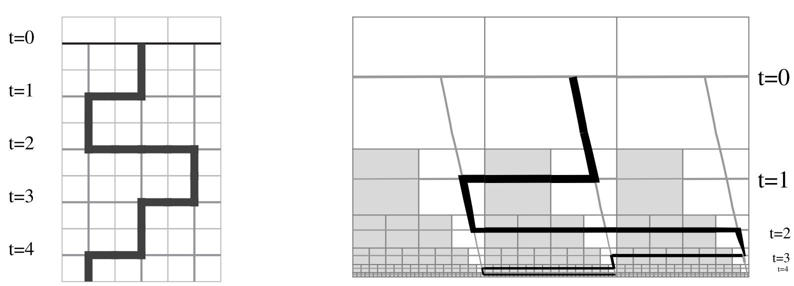

Figure 2: Rough sketch of Robinson's modeling of Wang's proof, in $\mathbb{H}^2$

He mentions that he was unable to show that the Domino Problem is undecidable as well. It is hardly surprising that this would be non-trivial to say the least— one cannot simply model Berger's proof in the same way that one models Wang's. There is no obvious way to create a useful hierarchical tiling in $\mathbb{H}^2$, in which the tiles clump into increasingly large patches, the patches of different sizes sharing some sort of useful structure. (We should add that there is not yet a meaningful, well-defined general idea of "hierarchical structure" that allows us to make the preceeding statement rigorously.) But more than this, it is heuristically clear that the old proof cannot be made to work at all: there is a sharp tension between the requirements imposed by a curved space and those imposed by allowing only finitely many tiles in the protoset. In particular "signals" of the sort used in the Berger/Robinson proof of the undecidability of the Domino Problem must travel over families of paths with varying curvature; these paths then will cut across the underlying regular tilings in infinitely many ways. There are ways around this, but these all seem to introduce roughly the same problem in slightly different form.)

From this and other considerations we make the following conjecture.

Conjecture 2.4 *In $\mathbb{H}^2$, among polygonal protosets, the Domino problem is decidable*

This second conjecture is really quite strange; in some sense it says that species of tilings in $\mathbb{H}^2$ are easier to understand than species of tilings in $\mathbb{E}^2$. But we will see further evidence for this conjecture as we proceed.

We should add that the Domino Problem is *not* decidable in $\mathbb{H}^n$, $n > 2$, by a simple trick: any protoset in $\mathbb{E}^{n-1}$ can be made into a protoset in $\mathbb{H}^n$; each tile in $\mathbb{E}^{n-1}$ becomes a prism bounded by horospheres in $\mathbb{H}^n$. With a suitable choice of construction, our new protoset in $\mathbb{H}^n$ will have to tile in layers, each of which corresponds to some tiling of $\mathbb{E}^{n-1}$ by the original protoset. Now since the Domino Problem is undecidable in $\mathbb{E}^{n-1}$, $n > 2$, the Domino Problem is undecidable in in $\mathbb{H}^n$. Note that the Domino Problem is decidable in $\mathbb{E}^1$.

3 Weak and Strong Aperiodicity

From the beginning, Wang recognized that, in the setting he considered, the Domino problem is decidable if every protoset that admitted a tiling admitted a tiling that has some compact fundamental domain, as described below. Berger thus showed there was a protoset that admitted a tiling but admitted no tiling with a compact fundamental domain. He explicitly gave such a protoset, though it famously contained over 20,000 tiles.

Over time such "aperiodic" tilings have been extensively studied. But the traditional definition that emerged— that a protoset is aperiodic iff it admits no tiling invariant under some translation— has proved inadequate: in general this has little to do with the original motivation for the study of aperiodicity, it allows examples that are quite unsatisfying, and in many settings, the definition simply makes no sense. We attempt to rectify this with the following definitions. The notion of "strong aperiodicity" apparently appears first in [30].

A tiling τ of X is to be called **weakly periodic** iff there exists an infinite cyclic subgroup H of $\mathcal{G}$ with $H\tau = \tau$ (i.e. for all $h \in H$, $h\tau = \tau$). A tiling that is not weakly periodic is said to be **strongly non-periodic**. A non-empty species $\Sigma(T)$ containing only strongly non-periodic tilings, and the corresponding protoset T, are said to be **strongly aperiodic**.

A tiling τ of X is to be called **strongly periodic** iff there exists a discrete subgroup H of $\mathcal{G}$ with X/H compact and $H\tau = \tau$. A tiling that is not strongly periodic is **weakly non-periodic**. A non-empty species $\Sigma(T)$ containing only weakly non-periodic tilings, and the corresponding protoset T, are said to be **weakly aperiodic**.

In $\mathbb{E}^2$, as long as the prototiles are not particularly pathological (cf. Theorem 3.7.1 of [16]), the three definitions of aperiodicity exactly coincide. But in general settings, the traditional definition is at best inadequate, and at worst meaningless:

The original motivation for the study of aperiodicity was the Domino Problem (Question 2.2). In particular, generalizing an observation[1] first made by Wang:

[1] In this paper we state several "observations". In effect these are templates for theorems; that is, they *are* theorems

Observation 3.1 *In any "nice" setting, if there is no weakly aperiodic protoset, then the Domino problem is decidable.*

By "nice" we mean that we require appropriate conditions so that if $\Sigma(T) = \emptyset$, there is some size disk that cannot be covered by any configuration of tiles in T, and that we can enumerate all configurations up to any given size.

These two conditions hold, for example, for species given by polygonal prototiles in $\mathbb{H}^n$ or $\mathbb{E}^n$, in which we require vertices to match to vertices, etc.

If no weakly aperiodic protosets exist, the algorithm to decide the Domino problem is simple: enumerate larger and larger configurations. One will eventually be stymied ($\Sigma(T) = \emptyset$) or one will come across a fundamental domain for a strongly periodic tiling.

Yet, neither the definition of weak aperiodicity nor the usual definition of aperiodicity seem to capture what are widely viewed, informally, as "legitimate" aperiodic tilings. For example, the tilings by Schmitt-Conway-Danzer tiles in $\mathbb{E}^3$ indeed have no translational symmetry, but these tiles do admit tilings that are invariant under a periodic screw motion.

Penrose [34] and others [5, 27] have constructed many examples of weakly aperiodic protosets in $\mathbb{H}^2$ (figure 3); these examples all make use of some imbalancing condition relying on the curvature of $\mathbb{H}^2$ that precludes the tiling of a compact quotient of $\mathbb{H}^2$. However, all of these examples do in fact admit a weakly periodic tiling and so are only weakly aperiodic. These examples are discussed further in Section 3.2 below.

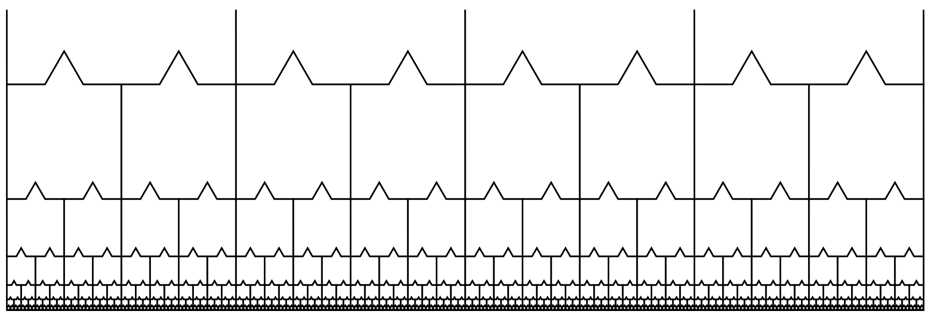

Figure 3: A weakly aperiodic einstein in $\mathbb{H}^2$

All of these "degenerate" examples are weakly but not strongly aperiodic. In short then, these definitions attempt to (a) return to the original motivation with the definition of weak aperiodicity and (b) strengthen and generalize the traditional notion with the definition of strong aperiodicity.

We then ask in each given specific setting $(X, \mathcal{G}, R)$:

Question 3.2 *Is there a weakly aperiodic protoset?*

Question 3.3 *Is there a strongly aperiodic protoset?*

Note again that in a "nice" setting, if the Domino Problem is undecidable, then there is indeed a weakly aperiodic protoset. And if every cocompact group acting on X has an infinite cyclic subgroup, as is true in $\mathbb{E}^n$, $\mathbb{H}^n$, then any strongly aperiodic protoset is weakly aperiodic. In the Euclidean plane (but not $\mathbb{E}^n, n > 2$, $\mathbb{H}^n$), if the tiles can meet in only finitely many ways, then the converse also holds, any weakly aperiodic protoset is strongly aperiodic.

We list several trends in the study of aperiodicity of species of tilings. Our list is narrowly focused: other, very important threads have emerged that we do not consider here, such as connections to ergodic theory (cf. [36, 37, 39, 40, 47], etc.) or to material science. Also *quasiperiodicity*, a particular model of non-periodicity, has become an extremely active and important area of study in its own right (cf. [2, 3, 24, 26, 28], etc).

in a wide variety of settings, but in any particular setting under consideration, it would be wise to check carefully that the observation does really hold.

3.1 Emergence of simple examples: the einstein problem

First, very small aperiodic protosets have emerged. Soon after Berger's result, R.M. Robinson gave a more easily understood example in the lovely [41]; Penrose [16] and then Amman [1] obtained an aperiodic protoset of just two tiles (allowing $\mathcal{G}$ to contain reflections and rotations). However these simple strongly periodic examples are far and few between; in [12], we give the first new example of an aperiodic pair in $\mathbb{E}^2$ in nearly twenty years. In [13] we give the only known strongly aperiodic pair in $\mathbb{E}^n$.

If we allow only translations, eight tiles will suffice in $\mathbb{E}^2$ [1, 12]; allowing only translations and square tiles with colored edges, as few as 13 tiles will do [21, 7]. In $\mathbb{H}^2$, there have been several examples of a weakly aperiodic monotile [34, 27]; there are generalizations to $\mathbb{H}^n$. There are also weakly aperiodic monotiles in $\mathbb{E}^n$ (the SCD of [43, 9]). However there is no known strongly aperiodic monotile (an "einstein") in any setting. For a fuller listing of small aperiodic protosets, see [12].

The following conjecture (or question) is well known:

Conjecture 3.4 *In $\mathbb{E}^2$, there is no weakly and no strongly aperiodic polygonal monotile (an "einstein").*

We go ahead and also state:

Conjecture 3.5 *In $\mathbb{E}^n$, $n > 2$ possibly restricting ourselves to letting $\mathcal{G}$ contain only translations, orientation preserving isometries, etc., with polyhedral tiles, possibly with markings, there is no strongly aperiodic monotile,*

There have been several near misses that shed some light on just for what a sharp theorem must account.

3.1.1 Small sets in $\mathbb{E}^2$

First, of course, as just discussed in 3.1 above and at length in [12], there are several *small* aperiodic protosets, containing as few as two tiles [1, 12, 13, 16, 33]. Moreover, it is quite easy to assure that only one tile accounts for arbitrarily much of the area in the tiling [12, 13, 35]. The Amman tile with Heesch number 3, the 8-isohedral tile of figure 6, and the hundreds of distinct monotiles in [16] suggest that the behaviour of even monohedral tilings is complex. Yet this does not seem to be enough; the conjectures above still seem correct.

3.1.2 Weakly aperiodic monotiles in $\mathbb{E}^n, n \geq 3$, $\mathbb{H}^n, n \geq 2$

These will be discussed in Section 3.2 below. But again, any proof of the above conjecture must somehow make use of the flat and restrictive nature of $\mathbb{E}^2$.

3.1.3 If "atlas"-style matching rules are allowed

Penrose has given a single tile that admits only aperiodic tilings— if certain non-standard matching rules are obeyed. In fact, this turns out to be quite general and trivial, as the following trivial-to-generalize theorem, shows:

Theorem 3.6 *Let T be a (possibly marked) protoset in $\mathbb{E}^2$ with $\mathcal{G}$ a square lattice of translations. Then there is a finite atlas $\mathcal{M}$ of allowed configurations of the dimer d (a 2×1 rectangle) such that $\Sigma(T)$ is mutually locally decomposable with the species $\Sigma(d, \mathcal{M})$ of dimer tilings.*

Here $\Sigma(d, \mathcal{M})$ is the set of tilings τ by d such that for each $gd \in \tau$, gd is in some allowed configuration hM, $M \in \mathcal{M}$, $h \in \mathcal{G}$.

The proof is trivial. Begin with any protoset T of tiles (four Wang tiles are shown at left in figure 4). In [15] we show how a broad class of meaningful edge-to-edge rules (eg. matching numbers, or colors, or bumps and nicks) are essentially equivalent up to mutual local derivability (in [13] we give a careful example of these techniques). In particular, we can assume we have "drawn" the prototiles— the points of the protiles have been colored black or white and coincident points on the boundaries of adjacent tiles are required to be colored the same way.

Clearly, in a wide and commonplace setting it is sufficient to "draw" our tiles to some finite resolution, with "pixels", colored black or white, of a given fixed size. By using other shapes of pixels, especially those arising from non-deterministic substitution species (that is, a system in which we can make the monotiles arbitrarily small and they can fill a particular shape of pixel in arbitrarily many ways), we can easily obtain similar results for tilings in which $\mathcal{G}$ is more general (eg. the pinwheel [36]). But note that this method will in general fail to work at all in curved spaces; in such a space, there often is a sharp limit on the density of information contained in monohedral tilings (cf. [10]).

A tiny portion of the top edge of the D tile is illustrated in the middle of figure 4. Each black pixel is split vertically into a pair of dimers; each white pixel is split horizontally (right, figure 4). For an atlas of allowed configurations of dimers, take these modified drawings of the Wang tiles. Now it should be clear that each tiling in $\Sigma(T)$ can be decomposed into a unique tiling by dimers satisfying this local rule; conversely, each tiling by dimers satisfying this local rule can be decomposed into a unique tiling in $\Sigma(T)$. Since there are aperiodic, drawable, protosets, there are aperiodic tilings by the dimer, with local matching rules. And of course, we expect that simpler, less wasteful, far more elegant examples can be found than that of figure 4.

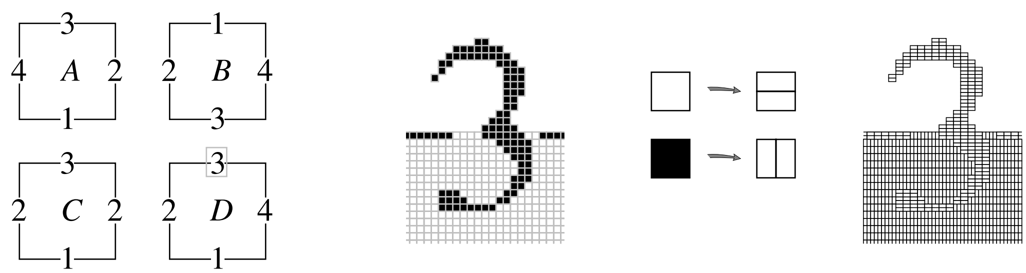

Figure 4: Every species is m.l.d. ("equivalent") to a species of dimer tilings with an "atlas"-style matching rule.

However, this trick is so pathetically cheap that this cannot be what we were really asking for. In the search for an einstein, our rules can only be that the coincident edges of neighboring tiles be compatible.

Nonetheless, the theorem is instructive, pointing out the need for a careful accounting of the complexity of a protoset with matching rules, as is carried out in [15].

3.1.4 Aperiodic shingles

In a very similar vein, Gummelt [17], Jeong and Steinhardt [20], and, most simply, Senechal [46] have shown that a single marked *shingle* can enforce the structure of the Penrose tilings. But this seems in the same spirit as 3.1.3, and we expect this generalizes considerably. In [15] we attempt to synthesize definitions of matching rules, shingles and markings to allow accounting for the true complexity of these constructions.

3.1.5 If similarities are allowed

It is unclear precisely what is involved in the loosening of the structure of X and $\mathcal{G}$. In particular, if we allow $\mathcal{G}$ to include scalings in $\mathbb{E}^2$ then a very large and uninteresting class of tiles admits monohedral tilings of all but a totally disconnected "singular" set.

Yet some of these tilings are still quite interesting. For example, the monotile of figure 1 admits only strongly non-periodic tilings if we impose certain restrictions on the topology of the singular set,[2] which seems to be a non-trivial statement. (The nature of any singular set plays a key role in any classification of species in which we allow similarities into $\mathcal{G}$.)

3.2 Emergence of different models

Just as aperiodic protosets have gotten smaller, a handful of general classes of aperiodic protosets has emerged. In each case, a certain kind of non-periodic structure is, somehow, magically, forced to arise. "Hierarchical" aperiodic species were examined first, by Berger [4]; then in [41, 33, 1, 29, 32, 36, 12], to name a few. This approach was finally reduced from art to science in [11], in which a systematic technique for creating such protosets is described.

There is a huge literature on "quasiperiodic" tilings, largely in physics journals ([2] gives a good mathematical entrypoint); Le [25] has given a general method for constructing aperiodic protosets of this sort. And a few special techniques in $\mathbb{E}^2$ have emerged, most notably that of Kari and Culik [21, 7].

Moreover, as mentioned above, special methods have produced weakly aperiodic protosets— even monotiles— in $\mathbb{H}^n$ (cf. [34, 27]) and $\mathbb{E}^n$ (cf. [43, 9]) and nonamenable spaces [5]. And Mozes has given a general technique for constructing strongly aperiodic protosets in large class of Lie groups[30].

The weakly aperiodic monotiles in $\mathbb{H}^2$ (which easily generalize to $\mathbb{H}^n$) all fail beautifully in $\mathbb{E}^2$:

Penrose's einstein makes use of a "Ponzi scheme"[3]— in the illustrated version of the tiles, each tile has one bump and two nicks. Any tiling of a compact quotient of $\mathbb{H}^2$ would have a finite number of tiles, yet have equal numbers of bumps and nicks. So no strongly periodic tiling by this tile exists; the tile is weakly aperiodic. This idea is generalized substantially in [5].

There can be no similar "Ponzi" construction in $\mathbb{E}^n$: a monotile with, say, more nicks than bumps admits no tiling at all, since in $\mathbb{E}^n$, the number of excess nicks in a configuration of radius r must be of order r^n; but on the other hand the number of tiles on the boundary, where these excess nicks will appear, is only of order r^{n-1}. At some point, one reaches a size beyond which no configurations can be constructed. Indeed, this is the basis of Amman's construction of a tile with Heesch number three [45].

The lovely construction in [27] makes use of a similar imbalance; the area of the tiles is incommensurate with π; hence the tile cannot tile any compact quotient of $\mathbb{H}^2$. On the other hand, there is no such restriction in Euclidean space.

Note that none of these constructions is at close to being a strongly aperiodic protoset in $\mathbb{H}^2$: in every case, these tiles do admit a tiling invariant under some infinite cyclic action.

The following conjecture is at once surprising and quite reasonable:

Conjecture 3.7 *There is no strongly aperiodic protoset in $\mathbb{H}^n$, $n \geq 2$.*

In an earlier circulated draft of this paper, we gave an even stronger conjecture; it appeared that the known weakly aperiodic protosets admit a configuration that (a) by itself has a compact fundamental domain and (b) is itself a (non-compact) fundamental domain. In figure 3 a row of tiles is such a configuration. That is, only a weak form of weak aperiodicity is achieved! (Incidentally, the SCD also satisfies this property).

Conjecture 3.8 *Every protoset T in $\mathbb{H}^2$ with $\Sigma(T) \neq \emptyset$ in fact has this "weak weak-aperiodicity".*

Lewis Bowen has since given a very nice counterexample to this conjecture [6]. The structure he gives can be detected by a semi-decidable procedure, as can weak-weak-aperiodicity, and so one may hope for a way to show Conjecture 2.4.

[2] Namely, that no subset of of the singular set is dense on the boundary of a closed disk in the plane.
[3] Named for the founder of a famous pyramid scheme in 1920's Boston

3.3 Complexity

It may be that no matter how many models emerge, certain aspects of aperiodicity will always remain mysterious. It seems to me, and perhaps to others, that because the Domino Problem is undecidable for polygonal tiles in $\mathbb{E}^2$, a meaningful categorization of all of the general kinds of aperiodicity, even in this setting, is likely to be impossible.

This is essentially due to a generalization of Observation 3.1. Recall that a semi-decidable problem is one for which there is an algorithm that will surely halt if the problem can be answered yes, but might not halt if the answer is no. So, the following properties are all semi-decidable: whether a "nice" protoset admits a strongly periodic tiling; whether a protoset does not admit a tiling at all; whether a protoset has the "weak weak-aperiodicity" described above.

Observation 3.9 *In any "nice" setting, suppose P is some property of a protoset that can be semi-decided by examining larger and larger configurations, then if every protoset T with $\Sigma(T) \neq \emptyset$ has property P, then the Domino problem is decidable.*

The algorithm is exactly as for Observation 3.1 above. Now the models of aperiodic behaviour found so far all seem to satisy some semi-decidable property. For example, though there is no adequately well-defined notion of "hierarchical aperiodicity" the known examples all work because large configurations act combinatorially like small configurations (the original prototiles). One can discover that a protoset admits configurations of this sort by examing larger and larger configurations (though one may not discover that a protoset *can't* admit such configurations). Whether a protoset admits such a hierarchical tiling is thus semidecidable.

Philosophically, the above observation has two consequences: first, aperiodic protosets of unimagined natures await us, even among Wang tiles in $\mathbb{E}^2$; second, this might always be the case.

As a practical matter, this may mean that there are relatively small sets that *are* aperiodic, but for *no particularly discernable reason*. For such a set, the Domino Problem would be "intractible"— that is, perhaps not mathematically undecidable, but at least unknowable to mortals, because one would be able to construct increasingly large configurations, apparently without rhyme or reason, without ever finding that the tiles admit no tiling or that they admit a periodic tiling. Kari reports that a set of 12 of the thirteen Wang tiles in [7] has proven to be intractible, so far [22]. And I know of a set of two polygonal tiles that have resisted analysis by hand for quite a while. There is reputed to be a set of four tiles in $\mathbb{E}^3$ that has defied analysis as well.

Even a monotile can give rise to complex behaviour— [16] contains illustrations of literally hundreds of distinctly interesting monotiles. As we see again and again in mathematics and the physical sciences, very simple local conditions can give rise to highly complex behaviour.

There are various projects underway to enumerate the behaviour of small classes of tiles, as best as possible. Of course, such a project can only partially succeed, and the goal would be to actually find small tile sets that are completely baffling.

If success is measured by how quickly one gets confused, these projects are sure to succeed:

Observation 3.10 *In a particular "nice" setting for which the Domino Problem is undecidable, suppose there are finitely many protosets of any given size. Define a function $f(n)$ to be the maximum, over all protosets of size n that are not weakly aperiodic, of the size of either the maximum configuration tiled (if the protoset does not tile at all) or the minimum fundamental domain (if the protoset tiles in a strongly periodic fashion). Then $f(n)$ has no computable bound— that is, for any computable function h, for any M, there is an $N > M$ with $f(N) > h(N)$.*

This is quite easy to see; if there were such a computable bound, one would know how far out one has to tile before being assured that the tiling can be completed.

It is very difficult to get one's mind around a function with no computable bound: the set of computable functions includes such gems as

$$h(1) := 1, \quad h(n) := n \underbrace{!\ldots!}_{h(n-1)}$$

The reader may be amused to work out the values of $h(2), h(3)$, etc. Of course for a *given* computable h it is unclear just when f will overtake h, but still one might expect rapid growth fairly quickly.

4 Generalizations of problems of Heesch, and Grünbaum and Shephard

Observation 3.10 above points to the connection, in any specific "nice" setting $(X, \mathcal{G}, R)$ between the decidability of the Domino Problem and bounds on the complexity of the behaviour of protosets that either don't tile X at all or admit a strongly periodic tiling. In fact, attempts to measure this complexity have appeared before in somewhat more restricted guise:

4.1 The Heesch number of a protoset

Given a configuration τ, we can define a **corona** of τ as a collection $C(\tau)$ of tiles so that

(a) $C(\tau) \cup \tau$ is a configuration;

(b) the support of τ is in the interior of the support of $C(\tau) \cup \tau$;

(c) and every tile in $C(\tau)$ is incident to τ.

We can inductively define an mth corona— a corona of a corona ... of τ. Let T be a protoset with $\Sigma(T) = \emptyset$. We define the **Heesch number** $H(T)$ of T to be the maximum m so that there is an mth corona of some tile in T. In a "reasonable" space X (cf. Thm. 3.8.1, [16]), such an m will exist, since if one can tile arbitrarily large regions, one can in fact tile the entire space. To simplify matters, if $\Sigma(T) \neq \emptyset$, we (non-standardly) set $H(T) = 0$. So the example of figure 5, found in an extensive computer search by J. Kari has Heesch number 2 [22].

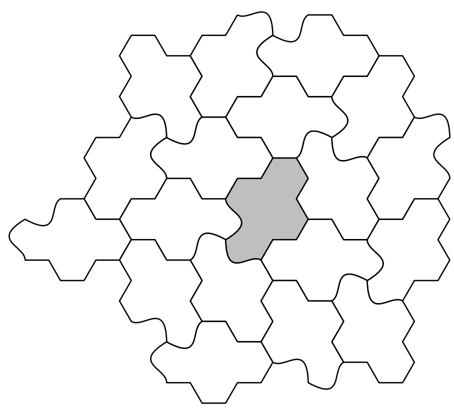

Figure 5: A new example of a tile with Heesch number 2 (J. Kari [22])

Heesch defined this number specifically for monotiles in $\mathbb{E}^2$ and asked, the following question in this setting [19]. We give the more general form of the question. In each specific setting we then ask,

Question 4.1 *(a) Is there a computable function $f(n)$ that bounds the Heesch number of each set of n prototiles? (b) In particular, if we restrict ourselves to monotiles, is the Heesch number bounded?*

As described above,

Observation 4.2 *In a "nice" setting, if such a computable bound exists then the Domino Problem is decidable.*

Currently, there are examples of single tiles with Heesch number up to three (cf. [45], p. 146), in $\mathbb{E}^2$, with polygonal tiles. But beyond this, almost nothing is known. In [16] we see

Conjecture 4.3 *[16] For polygonal monotiles in $\mathbb{E}^2$, the Heesch number is bounded.*

We also believe however, that

Conjecture 4.4 *For protosets of polygonal tiles in $\mathbb{H}^2$, the function $f(n)$ described above has a computable bound*

In essence, this says that once one can tile so far in $\mathbb{H}^2$, one can continue, pushing all bad behavior off to infinity. This conjecture is of a piece with conjectures 2.4, 3.7 and has been verified in an extremely limited context by the author's student, C. Mann.

4.2 The Isohedral number of a protoset

Given a strongly periodic tiling τ by prototiles T, we say τ is **n-isohedral**[4], with **isohedral number n**, for $n = \max_{A \in T} o(A)$, where $o(A)$ is the number of orbits of a prototile $A \in T$ under the largest subgroup of $\mathcal{G}$ leaving τ invariant. A 1-isohedral tiling is simply said to be isohedral. So for example, the strongly periodic tiling suggested in figure 6 is 8-isohedral.

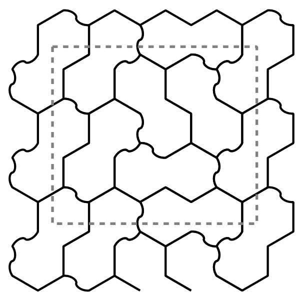

Figure 6: A tile with isohedral number 8 (J. Kari [22])

We say a protoset T is **n-isohedral**, with **isohedral number** n when $n = \min_{\tau \in \Sigma(T)} n_\tau$ where n_τ is the isohedral number of $\tau \in \Sigma(T)$. Note that n is defined and finite iff $\Sigma(T)$ is neither weakly aperiodic nor empty. To simplify Question 4.5, in either of these cases, we set $I(T) = 0$. The tiles in figure 6, found during an extensive computer search by J. Kari, are 8-isohedral.

Now there is some literature on n-isohedral *tilings* but there seems to be very little on isohedral *protosets*. We may have missed an article, but it appears, from a search of the literature, that the protoset of figure 6 has the highest known isohedral number [22] (during his computer search, Kari found several other tiles with isohedral number 8).

We ask, again, in each specific setting:

Question 4.5 *(a) Is there a computable function $g(n)$ that bounds the isohedral number of each set of n prototiles? (b) In particular, if we restrict ourselves to monotiles, is the isohedral number bounded?*

Here there is a less obvious logical connection to the decidability of the Domino Problem and the existence of weakly aperiodic protosets. The following observation, which motivates our final question, is due to J. Kari [22].

Observation 4.6 *In a "nice" setting, consider the "Period Problem": whether a given protoset T admits a strongly periodic tiling. Of course the Period Problem is at least semi-decidable. Now if there is a computable bound on the function $g(n)$ described above, then the Period Problem is*

[4]Note this is not the same definition as in [16]— There the total number of orbits are counted; here we take the maximum number of orbits of each prototile.

decidable. Note too that if the Period Problem is undecidable, then there exists a weakly aperiodic protoset. Consequently, if there is no weakly aperiodic protoset then the decidability of the Domino and Period Problems are (trivially) equivalent.

This is pretty clear: if there is a computable bound on g, then we have an algorithm to decide the Period Problem: examine configurations up to this bound. If a strongly periodic tiling exists with the given protoset, we will find it before this bound is reached.

Now suppose there is no weakly aperiodic protoset. Then any tiling which tiles at all does admit a periodic tiling. As not tiling at all is semi-decidable, in a "nice" setting, if there is no weakly aperiodic protoset, then the Period Problem is decidable.

The last part of the observation is tautological, but worth explicitly stating.

Question 4.7 *In a given setting, is the Period Problem decidable?*

Conjecture 4.8 *In* $\mathbb{E}^n$, $n \geq 2$, *with arbitrary polygonal protosets, the Period Problem is undecidable and there is no computable bound on* g.

Conjecture 4.9 *In* $\mathbb{E}^n$, $n \geq 2$ *with a monotile, the Period Problem is decidable and the isohedral number is bounded.*

Conjecture 4.10 *In* $\mathbb{H}^2$ *with arbitrary polygonal protosets the Period Problem is decidable and there is a computable bound on* g.

5 Summary

It would be wise to summarize our questions, observations, many known results and the conjectures. In the figure below, we give various implications that hold in "nice" settings.

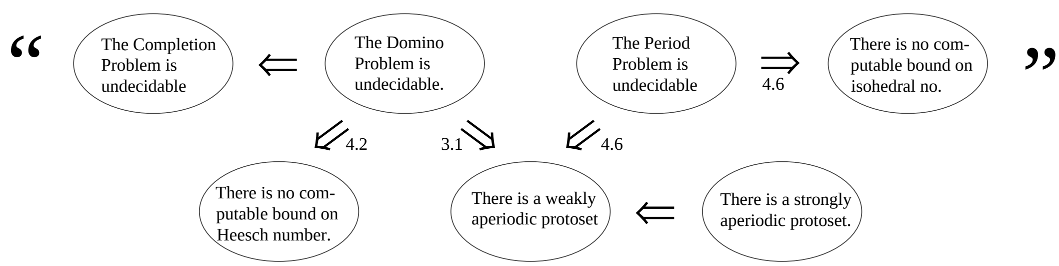

Figure 7: Various implications in a "nice" setting.

Summary of Results and Conjectures

Question	In $\mathbb{E}^2$:	In $\mathbb{E}^n$, $n \geq 3$:	In $\mathbb{H}^2$:	$\mathbb{E}^2$, *monotiles*:
(2.1) Comp. Prob. decidable?	No [48]	No, folk thm.	No [42]	*Conj 2.3: Yes*
(2.2) Domino Prob. decidable?	No [4, 41]	No, folk thm.	*Conj 2.4: Yes*	*Conj 2.3: Yes*
(3.2) ∃ weakly ap. protoset?	see below	SCD [43, 9]	[34, 27, 5]	*Conj 3.4: No* (folk)
(3.3) ∃ strongly ap. protoset?	[4, 41, 33, 1, 29, 32, 36, 12, 11], etc.	a few examples, including [11, 13, 23, 44, 7]	*Conj 3.7: No*	*Conj 3.4: No* (folk)
(4.1) ∃ comp. bound on Heesch?	No (follows from [4, 41])	No (folk)	*Conj 4.4: Yes*	*Conj 4.3: Yes*[16]
(4.5) Period Prob. decidable?	*Conj 4.8: No*	*Conj 4.8: No* in general, *Conj 4.9:* Yes for mono.	*Conj 4.10: Yes*	*Conj 4.9: Yes*
(4.7) ∃ comp. bound on Isohed.?	*Conj 4.8: No*	*Conj 4.8: No* in general, *Conj 4.9: Yes* for mono.	*Conj 4.10: Yes*	*Conj 4.9: Yes*

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Chaim Goodman-Strauss
Dept. of Mathematics
University of Arkansas
Fayetteville, AR 72701

cgstraus@comp.uark.edu