# A Strongly Aperiodic Set of Tiles in the Hyperbolic Plane

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**Abstract** We construct the first known example of a *strongly aperiodic* set of tiles in the hyperbolic plane. Such a set of tiles does admit a tiling, but admits no tiling with an infinite cyclic symmetry. This can also be regarded as a "regular production system" [5] that does admit bi-infinite orbits, but admits no periodic orbits.

# 1 Introduction

In any given fixed setting, such as tilings by polygonal tiles in the hyperbolic plane, we can examine several related questions, discussed much more completely in [4]:

*Is the "Completion Problem" undecidable?* That is, is there an algorithm to decide, given a set of tiles— a "protoset"— and some bounded configuration of copies of these tiles, whether the configuration can be extended to a tiling of the entire space by copies of these tiles.

*Is the "Domino Problem" undecidable?* That is, is there an algorithm to decide, given a protoset, whether the protoset admits a tiling of the space at all.

*Is there a "weakly aperiodic" protoset?* Such protosets do admit tilings, but no tiling with a compact fundamental domain. However, they may admit tilings that have an infinite cyclic symmetry (a period).

*Is there a "strongly aperiodic" protoset?* Such protosets admit tilings, but admit no tilings with even an infinite cyclic symmetry. We can go further and ask for protosets admitting tilings but admitting none with any non-trivial symmetry whatsoever.

A setting is "nice" if in any given tiling, there can be at most finitely many configurations of any bounded size. So for example, requiring tiles to be combinatorial polygons in a metric space is "nice". In any given nice setting, if the Domino Problem is undecidable, then so too is the Completion Problem, and there must exist a weakly aperiodic protoset.

However, weak aperiodicity is indeed weak; strong aperiodicity seems to fit many people's immediate sense of what aperiodicity should mean.

These problems were first studied in the Euclidean plane; Wang showed the Completion Problem is undecidable in that setting [13], and Berger answered the other three problems affirmatively [1]. In that setting, weak and strong aperiodicity coincide, for any protoset admitting tiling with infinite cyclic symmetry also admits a tiling with a compact fundamental domain (cf [6]). But in more general settings this is not the case.

In the hyperbolic plane, R. Robinson showed the Completion Problem is undecidable in [12]. He apparently spent some time attempting to show that the Domino Problem is undecidable, without success. About the same time, Penrose gave a weakly aperiodic protosets in the hyperbolic plane [10]; a very similar prototile is shown, in the upper half-plane model, in Figure 1. Note that the tile *does not* admit a tiling with a compact fundamental domain; yet it *does* admit a tiling with a period, an infinite cyclic symmetry. The tile is *weakly* but not *strongly* aperiodic.

As an aside, L. Sadun has pointed out that this example generalizes nicely; every symbolic substitution system corresponds to a weakly aperiodic set of tiles; given such a system, the periodic tilings admitted by the corresponding tiles precisely correspond to the periodic orbits in the space of bi-infinite words with "floating decimal point", under the substitution.

Since then, many further, but essentially similar, examples of weakly aperiodic protosets have been found, by Block and Weinberger [2], Margulis and Mozes [8] and others. Mozes has given an elegant construction of strongly aperiodic protosets in a large class of Lie groups [9].

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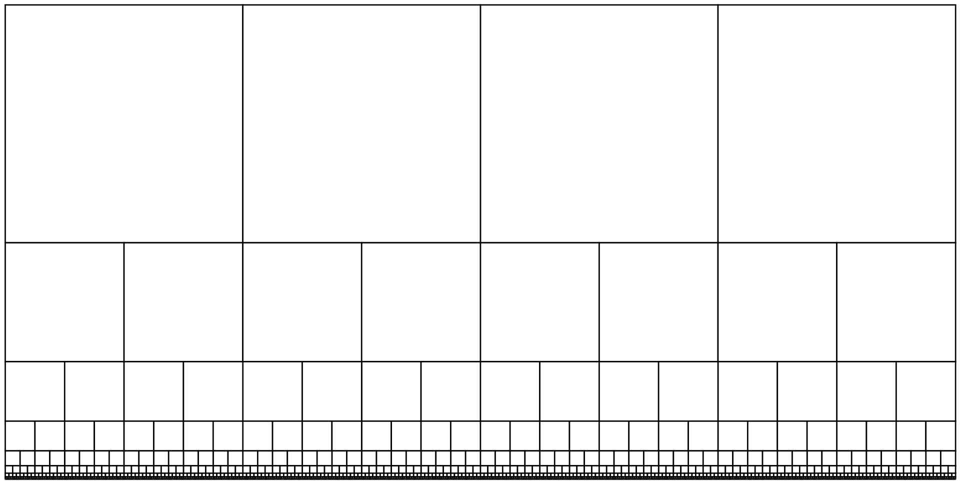

Figure 1: A tiling by a "weakly aperiodic" prototile in $\mathbb{H}^2$. Of the uncountable set of tilings admitted by this prototile, countably many have an infinite cyclic symmetry, but no tiling by this prototile has a symmetry with a compact fundamental domain [10].

However, the undecidability of the Domino Problem and the existence of a strongly aperiodic set of tiles have remained open in the hyperbolic plane. Here we answer the latter; the former is still unknown.

We can foliate $\mathbb{H}^n$ into parallel (meeting at the same point at infinity) horospherical leaves isometric to $\mathbb{E}^{n-1}$ and a transverse family of parallel geodesic leaves, each isometric to $\mathbb{H}^2$. As the Domino Problem is undecidable in $\mathbb{E}^{n\geq 2}$, it is not hard to show it is undecidable in $\mathbb{H}^{n+1}$. Similarly, once we have our construction of a strongly aperiodic protoset in $\mathbb{H}^2$, we can construct a strongly aperiodic protoset in $\mathbb{H}^{n\geq 3}$ .

Finally, in [5] we discuss "regular production systems"; these are a generalization of symbolic substitution dynamical systems to a model more suitable for discussing tilings in general. These regular production systems appear to be quite subtle. The construction here can be viewed as giving a regular production system on which there do exist (uncountably many) bi-infinite orbits but there exist no periodic orbits. This is in sharp contrast to symbolic substitution systems.

### 1.1 Acknowledgements

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## 2 Preliminaries

The construction is related to a construction due to J. Kari in the Euclidean plane [7]. The essential idea is to emulate a fixed-point free map $f$ acting on an interval of real numbers. Our tiles will be decorated relatives of the tile in Figure 1. Rows in the tiling will somehow correspond to real numbers. A row corresponding to a real number $\alpha$ will be able to fit above a row corresponding to real number $\beta$ if and only if $f(\alpha) = \beta$. Consequently, since $f$ is fixed-point free, no row can ever be repeated.

There are of course a few complications, but this gives the essential idea.

Throughout, we'll make particular choices, for $f$, for the interval on which $f$ acts, etc. None of these is particularly critical and the construction can be generalized to give an infinite family of relatively distinct prototiles.

## 2.1 The underlying non-periodic map on $\mathbb{S}^1$

For any real $a, b$ with $a < b$ let $[\![a,b]\!] = [a,b]/(a \sim b) \cong \mathbb{S}^1$. We first discuss an automorphism $f$ on $[\![1,2]\!]$. Let

$$f(\alpha) := \frac{4}{3}\alpha \quad \text{for } \alpha \in [1, \frac{3}{2}]$$
$$f(\alpha) := \frac{2}{3}\alpha \quad \text{for } \alpha \in [\frac{3}{2}, 2]$$

Note that $f$ is a well-defined bijection on $[\![1,2]\!]$. If we regard $f$ as a map on $[1,2]$, then $f$ is one-to-one with the exception that $f(1) = f(2)$, and well-defined except that $f(3/2) = 1$ and $f(3/2) = 2$. Here is a trivial but very important lemma:

**Lemma 1.** *For any $n = 1, 2\ldots$ and any $\alpha \in [\![1,2]\!]$, $f^k(\alpha) \neq \alpha$; that is, the action of $f$ on $[\![1,2]\!]$ has no finite orbits.*

**Proof** $f^n(\alpha) = 2^m/3^n\alpha$ for some $m$, $n \leq m \leq 2n$. If $2^m/3^n\alpha = \alpha$, then $\alpha = 0$ or $m = n = 0$. □

**Lemma 2.** *Every $\alpha \in [\![1,2]\!]$ has a unique orbit $\{f^i(\alpha), i \in \mathbb{Z}\}$. For every $\alpha \in [\![1,2]\!]$, this orbit is dense in $[\![1,2]\!]$.*

**Proof** Each $\alpha$ lies in a unique bi-infinite orbit since $f$ is a bijection.

Let addition act on $[\![0,1]\!]$. Since $\log_2 3$ is irrational, for any $\beta \in [\![0,1]\!]$, the set $\{(\beta - n\log_2 3) \mid n \in \mathbb{Z}\}$ is dense in $[\![0,1]\!]$. Now note that $\log_2 : [\![1,2]\!] \to [\![0,1]\!]$ is a well defined homeomorphism. Let $\alpha \in [\![1,2]\!]$. For any $n \in \mathbb{Z}$, there exists $m \in \mathbb{Z}$ such that $f^n(\alpha) = 2^m/3^n\ \alpha$. Since $f^n(\alpha) \in [\![1,2]\!]$:

$$\log_2 f^n(\alpha) = m + \log_2 \alpha - n\log_2 3 = \log_2 \alpha - n\log_2 3$$

Consequently, the set $\{\log_2 f^n(\alpha)\}$ is dense in $[\![0,1]\!]$ and so $\{f^n(\alpha)\}$ is dense in $[\![1,2]\!]$. □

## 2.2 Balanced sequences

We will be encoding real numbers in the interval $[1,2]$ as doubly-infinite sequences of 1's and 2's, and the action of $f$ as certain productions on these sequences. These productions will have to be "expansive", in order to correspond to tilings of the hyperbolic plane, which will lead to complications.

Consider any finite sequence $\omega = \{\omega_1, \ldots, \omega_n\}$ of integers. The **average** $\overline{\omega}$ of $\omega$ is simply

$$\overline{\omega} := \frac{1}{n}\sum_1^n \omega_i$$

Doubly infinite sequences $\omega = \{\omega_n\}_\mathbb{Z} \subset \mathbb{Z}^\mathbb{Z}$ of integers rarely have any sort of well-defined average. However, we can define:

A doubly infinite sequence $\omega$ of integers has **average** $\alpha$ with **maximum imbalance** $N \in \mathbb{N}$ if and only if (a) for every $m \in \mathbb{Z}$, $n \in \mathbb{N}$, the sequence $|\overline{\{\omega_m, \ldots \omega_{n+m}\}} - \alpha| < N/n$ and (b) $N$ is the smallest natural number for which (a) holds. An infinite sequence is **imbalanced** if it has no average.

A doubly infinite sequence $\omega$ is **balanced, with average** $\alpha$ if and only if $\omega$ has average $\alpha$ with maximum imbalance 1. Trivially:

**Lemma 3.** *If $\omega$ is balanced with average $\alpha \in \mathbb{R}$, then each $\omega_n \in \{\lfloor\alpha\rfloor, \lceil\alpha\rceil\}$*

Of course if $\alpha \in \mathbb{Z}$, $\lfloor\alpha\rfloor = \lceil\alpha\rceil = \alpha$. From time to time we can simplify discussion if we take, for any given doubly infinite sequence $\omega = \{\omega_n\}$ of reals:

$\Sigma_0\omega = 0$
For $n > 0$, $\quad \Sigma_n\omega = \sum_{i=1}^{n} \omega_i$
For $n < 0$, $\quad \Sigma_n\omega = -\sum_{i=n+1}^{0} \omega_i$

Note that for any $n$, positive, negative or zero, $\quad \Sigma_n - \Sigma_{n-1} = \omega_n$ We now define, given any $\alpha$ and a parameter $t \in \mathbb{R}$, a doubly infinite sequence $\omega(\alpha, t) \in \mathbb{Z}^{\mathbb{Z}}$. We'll then show that sequences described in this fashion are *exactly* the balanced sequences, with average $\alpha$. For each $n \in \mathbb{Z}$, let

$$\omega_n = \omega(\alpha, t)_n = \lfloor n\alpha + t\rfloor - \lfloor (n-1)\alpha + t\rfloor$$

For any $\alpha \in \mathbb{R}, t \in \mathbb{R}$ and $k \in \mathbb{Z}$, note also that

$$\omega(\alpha, t) = \omega(\alpha, t + k)$$

so we may, as convenient, assume $t \in [0, 1)$. Note that for all $n$ (positive, negative or zero)

$$\Sigma_n\omega = \lfloor n\alpha + t\rfloor \tag{1}$$

There is a simple interpretation of $\omega(\alpha, t)$: Consider the line in the Cartesian plane with slope $\alpha$ and $y$-intercept $t$. Then consider the "closest staircase" beneath this line, that is, the path of edges on the integer lattice that most closely approximates this line without exceeding it. Then at each $x$-value $n$, $\omega_n$ is the height of the riser, $\omega$ is the sequence of these heights, and $\Sigma_n\omega$ is the cumulative height of the staircase.

**Lemma 4.** *For any $\alpha, t \in \mathbb{R}$, the sequence $\omega(\alpha, t)$ is balanced.*

**Proof** First, consider any $n$ consecutive digits in any $\omega(\alpha, t)$. The average of these digits is, for some index $m$:

$$\frac{1}{n}\sum_{i=m+1}^{m+n} \omega_i = \frac{1}{n}\left(\lfloor (m+n)\alpha + t\rfloor - \lfloor m\alpha + t\rfloor\right)$$

Using the identity $r - s - 1 < \lfloor r\rfloor - \lfloor s\rfloor < r - s + 1$ for any real $r, s$,

$$\frac{n\alpha - 1}{n} < \frac{1}{n}\left(\lfloor (m+n)\alpha + t\rfloor - \lfloor m\alpha + t\rfloor\right) < \frac{n\alpha + 1}{n}$$

and

$$\left|\frac{1}{n}\left(\lfloor (m+n)\alpha + t\rfloor - \lfloor m\alpha + t\rfloor\right) - \alpha\right| < \frac{1}{n}$$

Therefore, any finite subsequence of length $n$ has average within $1/n$ of $\alpha$ and the sequence is balanced. $\square$

**Lemma 5.** *Conversely, given any balanced word $\omega$, with average $\alpha$, there is some $t \in [0, 1)$ with $\omega = \omega(\alpha, t)$.*

**Proof** Let $\omega$ be balanced with average $\alpha$. We now pin down the possible values of $t$: for each $m \in \mathbb{Z}$, briefly define

$$I_m = \{t \in [0, 1) \mid \lfloor m\alpha + t\rfloor = \Sigma_m\omega\}$$

These are the values of $t$ so that $\Sigma_m\omega = \Sigma_m\omega(\alpha, t)$. All we need is a point in the intersection of all these $I_m$'s. A little manipulating of notation yields that each

$$I_m = [\Sigma_m\omega - m\alpha, \Sigma_m\omega - m\alpha + 1) \cap [0, 1)$$

and in particular is a half-open interval. Suppose $\cap I_m = \emptyset$; then since the $I_m$ are half-open intervals, there is a disjoint pair $I_i, I_j$, $i < j$. But then the average of the sequence $\{\omega_{i+1}, \dots, \omega_j\}$ is at least $1/(j-i)$ from $\alpha$. So $\cap I_m \neq \emptyset$. Let $t$ be any value in this intersection and one sees $\omega = \omega(\alpha, t)$. $\square$

Again though, by far most sequences, even with digits restricted to two consecutive integers, are not balanced.

### 2.3 A map on balanced sequences

Our basic strategy is to encode each $\alpha \in [\![1, 2]\!]$ as a balanced sequence in $S := \{1, 2\}^{\mathbb{Z}}$; we would then like to define an expansive substitution system taking balanced sequences, with average $\alpha$ say, to balanced sequences with average $f(\alpha)$. Unfortunately this does not seem possible. (In [7], the substitution was not expansive and there were no difficulties.)

Here we give one method of substituting that *almost* achieves our goal. In the next section, we will modify this method and produce a "regular production system" that will satisfy our needs. Our discussion here will be informal, since we'll have to redefine everything soon anyway. Take two sets of sequences in $S$:

$$S_1 = \{\omega \mid \forall n \in \mathbb{Z},\ \text{if } \omega_n = 2,\ \text{then } \omega_{n+1} \neq 2\} \subset S$$

$$S_2 = \{\omega \mid \forall n \in \mathbb{Z},\ \text{if } \omega_n = 1,\ \text{then } \omega_{n+1} \neq 1\} \subset S$$

Any finite subsequence of a doubly infinite sequence in $S_1, S_2$ has average in $[1, \frac{3}{2}], [\frac{3}{2}, 2]$. And though not every sequence in $S_1 \cup S_2$ is balanced, every balanced sequence in $S$ is in $S_1 \cup S_2$. Informally, define a substitution $\sigma$ taking $S_1 \cup S_2$ to $S$ by the rules:

$1 \to 112 \quad 12 \to 222222 \quad \text{for } \omega \in S_1$
$2 \to 112 \quad 12 \to 111111 \quad \text{for } \omega \in S_2$

As each sequence in $S_1$, ($S_2$) can be uniquely partitioned into subsequences 1 and 12 (12 and 1), these rules describe well-defined maps from $S_1 \cup S_2$ to $S$ (at least on finite subsequences— we have to set some conventions for a proper definition of the action on infinite sequences).

Note that on finite sequences, these substitutions expand the length of the sequence by a factor of 3 and if the average of the finite sequence is $\alpha$, the average of the new sequence will be $f(\alpha)$. For example $121112 \to 222222\ 112\ 112\ 222222$. The average of 121112 is $\frac{4}{3}$ and the average of 222222112112222222 is $\frac{16}{9} = f(\frac{4}{3})$.

The construction would be simpler if this map took balanced sequences to balanced sequences (and even better, if balanced words themselves formed a regular language; see below). But alas this is not the case. Consider, for example, an extreme example: $\dots(1)^{12}2\dots$ which is taken to the quite imbalanced $\dots(112)^{12}222222\dots$.

Happily, however, balanced sequences are *almost* taken to balanced sequences, and this is enough to make our construction work. In the example above, if we can just interchange a few 1's and 2's, we can get the balanced sequence $\dots(1212112)^6\dots$, as shown in Figure 2. We will continue this example below.

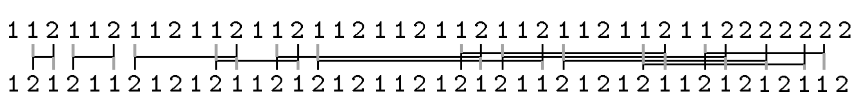



Figure 2: Rebalancing $\sigma(1^{12}12))$

This figure actually illustrates just about the worst possible case, an instance in which the greatest number of pairs must be exchanged across a given spot. It turns out that if we are allowed to interchange 1's and 2's across arbitrary distances, and as many pairs as we please, *but* at most four pairs are interchanged across any given spot, we can balance the image under this map of any balanced sequence $\omega$.

# 3 The Construction

In order to make our ideas precise, we define a "regular production system" that correctly encodes the action of $f$ on balanced sequences. We first pause for general definitions; in Section 3.2 we define our system. In Section 3.3 we obtain the key Theorem 9. In Section 4 we then construct a set of tiles in $\mathbb{H}^2$ that encode the regular production system and show that these are strongly aperiodic.

## 3.1 Regular production systems

We use [3] for standard definitions regarding languages. The following is taken from [5]. Let $\mathcal{A}$ be any finite alphabet and $\mathcal{L} \subset \mathcal{A}^*$ be any regular language on $\mathcal{A}$. Generally $\mathcal{L}$ will be regular; here we take this to mean that $\mathcal{L}$ corresponds to the set of paths in some directed graph.

For any word $\mathtt{W}$, let $\|\mathtt{W}\|$ be the length of $\mathtt{W}$. We define the language $\mathcal{L}^\infty \subset \mathcal{A}^{\mathbb{Z}}$ of **infinite words** to be sequences $\mathtt{W} \in \mathcal{A}^{\mathbb{Z}}$ such that every finite subsequence $\mathtt{W}(i) \dots \mathtt{W}(j)$ is a subword of some word in $\mathcal{L}$. In general, $\mathcal{L}^\infty$ may be empty. However, if $\mathcal{L}$ is an infinite regular language, by the Pumping Lemma (cf. any standard reference), then $\mathcal{L}^\infty \neq \emptyset$. Let $\zeta : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ be the usual shift map, $(\zeta(\mathtt{W}))(i) = \mathtt{W}(i-1)$.

Given an infinite set $\{\mathtt{V}_n\}_{n\in\mathbb{Z}} \subset \mathcal{A}^*$ a word $\mathtt{W} \in \mathcal{A}^{\mathbb{Z}}$ is the **infinite concatonation** $\dots \mathtt{V}_{-1}\mathtt{V}_0\mathtt{V}_1 \dots$ iff for all $n$ $\mathtt{W}(a_n) \dots \mathtt{W}(a_{n+1}-1) = \mathtt{V}_n$ where $a_n = \Sigma_n \|\mathtt{V}_i\|$. This definition coincides with what one might expect.

A **production relation** $\mathcal{R} \subset (\mathcal{L} \times \mathcal{L}) \cup (\mathcal{L}^\infty \times \mathcal{L}^\infty)$ satisfies:

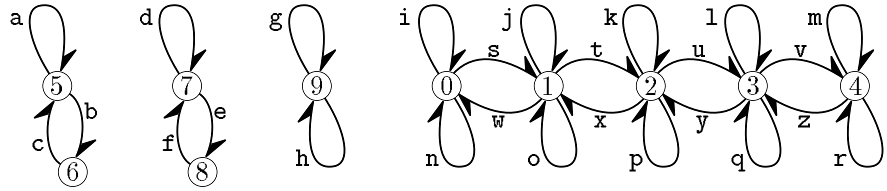
1. There is a finite set $\mathcal{R}_0 \subset (\mathcal{A} \times \mathcal{L})$ of "replacement rules", and $\mathcal{R}_0 \subset \mathcal{R}$.
2. For any $\mathtt{W}, \mathtt{V} \in \mathcal{L}$, $(\mathtt{W}, \mathtt{V}) \in \mathcal{R}$ if and only if there exists $\{\mathtt{V}_i\}_{\|\mathtt{W}\|} \subset \mathcal{L}$ with $(\mathtt{W}(i), \mathtt{V}_i) \in \mathcal{R}_0$ and $\mathtt{V} = \mathtt{V}_1 \dots \mathtt{V}_{\|\mathtt{W}\|}$.
3. For any $\mathtt{W}, \mathtt{V} \in \mathcal{L}^\infty$, $(\mathtt{W}, \mathtt{V}) \in \mathcal{R}$ if and only if there exist $\{\mathtt{V}_i\}_{\mathbb{Z}} \subset \mathcal{L}$ and natural $j$, $0 \le j < \|\mathtt{V}_0\|$ such that for all $i \in \mathbb{Z}$, $(\mathtt{W}(i), \mathtt{V}_i) \in \mathcal{R}_0$ and $\zeta^j(\mathtt{V}) = \dots \mathtt{V}_{-1}\mathtt{V}_0\mathtt{V}_1 \dots$.

For $(\mathtt{W}, \mathtt{V}) \in \mathcal{R}$, we will write $\mathtt{W} \to \mathtt{V}$ and say "$\mathtt{W}$ produces $\mathtt{V}$". Though the notation suggests that the relation is a function, it is not: a given word may be related to one, several, or no other words. A **regular production system** $(\mathcal{A}, \mathcal{L}, \mathcal{R})$ is specfied by an alphabet $\mathcal{A}$, regular language $\mathcal{L}$ on $\mathcal{A}$ and production relation $\mathcal{R}$ on $\mathcal{L} \cup \mathcal{L}^\infty$.

An **orbit** in a production system $(\mathcal{A}, \mathcal{L}, \mathcal{R})$ is any set $\{(\mathtt{W}^i, j_i)\}_{i\in\mathbb{Z}} \subset \mathcal{L}^\infty \times \mathbb{N}$ such that for all $i \in \mathbb{Z}$, $(\mathtt{W}^i, \mathtt{W}^{i+1}) \in \mathcal{R}$, with shift $\zeta^{j_i}$. An orbit is **periodic** if and only if there is some $n \ge 1$ with $\mathtt{W}^i = \mathtt{W}^{i+n}$, $j_i = j_{i+n}$ for all $i$, and the period of such an orbit is the minimal such $n$.

## 3.2 A regular production system

We now consider the following regular production system. Let $\mathcal{A} = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \dots, \mathtt{x}, \mathtt{y}, \mathtt{z}\}$. The language $\mathcal{L}$ is given by the graphs




Note that $\mathcal{L}$ is the disjoint union of the sublanguages $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4$ given by the left, left-middle, right-middle and right components of the graph, respectively. So, for example, $\mathcal{L}_1 = \mathcal{L} \cap \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}^*$. Let $\mathcal{L}_{12} = \mathcal{L}_1 \cup \mathcal{L}_2$. The labels $0, 1, \dots, 9$ on the vertices will be used later.

The 78 productions are

$$\mathtt{a},\mathtt{d} \to \mathtt{ggh} \qquad \mathtt{b},\mathtt{c} \to \mathtt{hhh} \qquad \mathtt{e},\mathtt{f} \to \mathtt{ggg}$$

$$\mathtt{g} \to \mathtt{i},\mathtt{j},\mathtt{k},\mathtt{l},\mathtt{m},\mathtt{s},\mathtt{t},\mathtt{u},\mathtt{v} \qquad \mathtt{i},\mathtt{j},\mathtt{k},\mathtt{l},\mathtt{m},\mathtt{w},\mathtt{x},\mathtt{y},\mathtt{z} \to \mathtt{a},\mathtt{b},\mathtt{e}$$

$$\mathtt{h} \to \mathtt{m},\mathtt{o},\mathtt{p},\mathtt{q},\mathtt{r},\mathtt{w},\mathtt{x},\mathtt{y},\mathtt{z} \qquad \mathtt{n},\mathtt{o},\mathtt{p},\mathtt{q},\mathtt{r},\mathtt{s},\mathtt{t},\mathtt{u},\mathtt{v} \to \mathtt{c},\mathtt{d},\mathtt{f}$$

Note that words in $\mathcal{L}_{12}$ produce only words in $\mathcal{L}_3$, which produce only words in $\mathcal{L}_4$, which in turn produce only words in $\mathcal{L}_{12}$.

We define numerical values for words in $\mathcal{L}_{12} \cup \mathcal{L}_3$ by a map $\overline{*}$; we set $\overline{\mathtt{a}} = \overline{\mathtt{b}} = \overline{\mathtt{e}} = \overline{\mathtt{g}} = 1$ and $\overline{\mathtt{c}} = \overline{\mathtt{d}} = \overline{\mathtt{f}} = \overline{\mathtt{h}} = 2$. For a word $\mathtt{W} \in \mathcal{L}_{12} \cup \mathcal{L}_3$, take $\overline{\mathtt{W}} = \overline{\{\overline{\mathtt{W}_i}\}} = \Sigma\overline{\mathtt{W}_i}/\|\mathtt{W}\|$. We say word $\mathtt{W}$ is balanced/imbalanced, etc., if and only if the sequence $\{\overline{\mathtt{W}_i}\}$ is as well. Note words in $\mathcal{L}_1$, $\mathcal{L}_2$ have averages in $[1, 3/2]$, $[3/2, 2]$ respectively.

A few examples would be helpful: The word $\mathtt{abc}$, with average $\frac{4}{3}$ produces only the word $\mathtt{g}^2\mathtt{h}^7$ with average $\frac{16}{9} = f(\frac{4}{3})$. The word $\mathtt{is}$ produces $\mathtt{bd}$ and $\mathtt{ef}$, but $\mathtt{iso}$ produces only $\mathtt{efd}$. There is tremendous flexibility in the productions of words in $\mathcal{L}_3$: $\mathtt{gh}$ already produces seventeen words. To continue the example of Section 2.3:

The word $\mathtt{W} := \mathtt{a}^{12}\mathtt{bc}$, with average $\frac{15}{14}$, produces $\mathtt{W}^1 := (\mathtt{ggh})^{12}\mathtt{h}^6$ with average $\frac{60}{42} = f(\frac{15}{14})$. In turn $\mathtt{W}^1$ produces 558,717,003,837 words! But among these we have

$$\mathtt{W}^2 := \mathtt{iswsjwsjojtxjtxtkpkkpkuyulyulqlvzlvzqypxwn} \tag{2}$$

which produces only $\mathtt{W}^3 := (\mathtt{bcbcabc})^6$ which is balanced, with average $\frac{10}{7}$.

One should think of words in $\mathcal{L}_1^\infty$ as (hopefully) balanced words with average in $[1, 3/2]$, words in $\mathcal{L}_2^\infty$ as (hopefully) balanced words with average in $[3/2, 2]$, words in $\mathcal{L}_3^\infty$ as the result of applying the substitutions $S_1$ and $S_2$ described in Section 2.3, and words in $\mathcal{L}_4^\infty$ as somehow accounting for the rebalancing that is required after applying the substitutions.

The proof of the following is trivial, but worth the reader's verifying:

**Lemma 6.** *Let $\mathtt{W} \in \mathcal{L}_{12}$ have average $\alpha$. If $\mathtt{W}$ produces $\mathtt{W}^1$ (in $\mathcal{L}_3$), $\mathtt{W}^1$ has average $f(\alpha)$. If in turn $\mathtt{W}^1 \in \mathcal{L}_3$ produces $\mathtt{W}^2$ (in $\mathcal{L}_4$) which produces $\omega^3$ (in $\mathcal{L}_{12}$) with average $\alpha'$, then $|\alpha' - f(\alpha)| \leq 4/\|\mathtt{W}^1\|$.*

Finally, define a relation $\mathcal{R}^3 \subset \mathcal{L}_{12}^\infty \times \mathcal{L}_{12}^\infty$ as: $(\mathtt{W}, \mathtt{W}') \in \mathcal{R}^3$ if and only if there are words $\mathtt{W}^1 \in \mathcal{L}_3 \cup \mathcal{L}_3^\infty$, $\mathtt{W}^2 \in \mathcal{L}_4^\infty$ with $\mathtt{W}$ producing $\mathtt{W}^1$ producing $\mathtt{W}^2$ producing $\mathtt{W}'$.

## 3.3 Orbits in our production system

We now set out to show that there do exist orbits in our regular production system, but that there exist no periodic orbits. In Section 4 we will then convert the system into a set of strongly aperiodic tiles in $\mathbb{H}$.

**Lemma 7.** *For any $\omega(\alpha, t)$, $\alpha \in [1, 2]$, there exists $\mathtt{W} \in \mathcal{L}_{12}^\infty$ with $\{\overline{\mathtt{W}_i}\} = \omega$; this $\mathtt{W}$ is unique unless $\alpha = \frac{3}{2}$.*

Denote this $\mathtt{W}$ by $\mathtt{W}(\alpha, t)$.

**Proof** If $\alpha \in [1, \frac{3}{2}]$, the sequence $\omega(\alpha, t)$ can be uniquely partitioned into sub-sequences $\{1\}$ and $\{1, 2\}$. Replace these with $\{\mathtt{a}\}, \{\mathtt{b}, \mathtt{c}\}$ to obtain the sequence of letters in $\mathtt{W} \in \mathcal{L}_1^\infty$. Similarly, if $\alpha \in [\frac{3}{2}, 2]$ we obtain a word $\mathtt{W} \in \mathcal{L}_2^\infty$. □

**Lemma 8.** *For any $\alpha, t$, we have $(\mathtt{W}(\alpha, t), \mathtt{W}(f(\alpha), st)) \in \mathcal{R}^3$, where $s = 3f(\alpha)/\alpha = 2, 4$.*

**Proof** Let $\mathtt{W}^3 = \mathtt{W}(f(\alpha), st)$. Now $\mathtt{W}^0 = \mathtt{W}(\alpha, t)$ produces exactly one word $\mathtt{W}^1$ (up to choice of shift, which we take to be trivial); $\mathtt{W}^1$ is in $\mathcal{L}_3^\infty$. Let $\omega^j = \{\overline{\mathtt{W}_i^j}\}$, $j = 0, 1, 3$. We will show that $\omega^1$ is "close to" $\omega^3$ by defining:

$$b_n = \Sigma_n \omega^3 - \Sigma_n \omega^1$$

**Claim:** $b_n = 0, 1, 2, 3$ or $4$. *Proof:* Let $N = \lfloor n/3 \rfloor$. From the definitions note that

$$s\lfloor \alpha N + t \rfloor = \Sigma_{(3N)}\omega^1$$

and that

$$s\lfloor \alpha(N+1) + t \rfloor = \Sigma_{(3N+3)}\omega^1$$

For any $x, a, b$ we have $x\lfloor a \rfloor + (1-x)\lfloor b \rfloor \le xa + (1-x)b < x\lfloor a \rfloor + (1-x)\lfloor b \rfloor + 1$ since $\lfloor a \rfloor \le a < \lfloor a \rfloor + 1$. Consequently for any $x$:

$$(1-x)\lfloor \alpha N + t \rfloor + x\lfloor \alpha(N+1) + t \rfloor \le \alpha(N+x) + t < (1-x)\lfloor \alpha N + t \rfloor + x\lfloor \alpha(N+1) + t \rfloor + 1$$

Multiplying through by $s$ we have

$$(1-x)\Sigma_{(3N)}\omega^1 + x\Sigma_{(3N+3)}\omega^1 \le s\alpha(N+x) + st < (1-x)\Sigma_{(3N)}\omega^1 + x\Sigma_{(3N+3)}\omega^1 + 1$$

Observing that for $x = 0, \frac{1}{3}, \frac{2}{3}$

$$\Sigma_{(3N+3x)}\omega^3 = \lfloor f(\alpha)(3N+3x) + st \rfloor = \lfloor s\alpha(N+x) + st \rfloor$$

that

$$\Sigma_{(3N+3x)}\omega^1 = \lfloor (1-x)\Sigma_{(3N)}\omega^1 + x\Sigma_{(3N+3)}\omega^1 \rfloor$$

and that if $a \le b < a + n$ where $n \in \mathbb{Z}$; then $\lfloor a \rfloor \le \lfloor b \rfloor \le \lfloor a \rfloor + n$ we now have:

$$0 \le b_n = \Sigma_{(3N+3x)}\omega^3 - \Sigma_{(3N+3x)}\omega^1 \le s \le 4$$

and the claim is proven.

**Claim:** $|b_n - b_{n-1}| \le 1$. *Proof:* Recalling that $\omega_n^3, \omega_n^1 \in \{1, 2\}$:

$$\begin{aligned} b_n - b_{(n-1)} &= \Sigma_n\omega^3 - \Sigma_n\omega^1 - \Sigma_{(n-1)}\omega^3 + \Sigma_{(n-1)}\omega^1 \\ &= \omega_n^3 - \omega_n^1 \\ &= -1, 0, 1 \end{aligned}$$

Now consider $\mathtt{W}^2$, defined by taking $\mathtt{W}_n^2$ to be the unique letter in $\mathcal{A}$ produced by $\mathtt{W}_n^1$, producing $\mathtt{W}_n^3$, describing the transition from state $b_{(n-1)}$ to $b_n$. Such a letter exists by Claim 1, and $\mathtt{W}^2 \in \mathcal{L}_4^\infty$ by Claim 2. We now have that $\mathtt{W}^0 \to \mathtt{W}^1 \to \mathtt{W}^2 \to \mathtt{W}^3$, or $(\mathtt{W}^0, \mathtt{W}^3) \in \mathcal{R}^3$. □

In particular, then:

**Theorem 9.** *There exist (uncountably many) distinct orbits in $\mathcal{L}^\infty$ under $\mathcal{R}$. However, there exist no periodic orbits in $\mathcal{L}^\infty$ under $\mathcal{R}$.*

**Proof** Let $\{\alpha_n\}$ be any orbit in $\mathbb{S}$ under $f$ (There are uncountably many such orbits, even modulo reindexing). Then $\{(\mathtt{W}(\alpha_n, 0), 0)\}$ is an orbit under $\mathcal{R}^3$ and consequently there exists a orbit in $\mathcal{L}^\infty$ under $\mathcal{R}$. (We have even more choices for $\{(\mathtt{W}(\alpha_n, t_n), k_n)\}$ parametrized by the choice of $t_0 \in [0, 1)$ and the sequence of shifts $\{k_n\}$.)

Now suppose there exists a periodic orbit $\{(\mathtt{W}^n, k_n)\}$; from the definition of the production, the period must be a multiple of 3, say $3M$. We may reindex so that $\mathtt{W}^{3n} \in \mathcal{L}_{12}^\infty$ for all $n$.

Consider $\{\omega^n\}$ where $\omega^n = \{\overline{\mathtt{W}_i^{3n}}\}$, and the sequence $\{\Delta_n\}$ where $\Delta_n = \frac{4}{3}$ if $\mathtt{W}^{3n} \in \mathcal{L}_1^\infty$ and $\Delta_n = \frac{2}{3}$ if $\mathtt{W}^{3n} \in \mathcal{L}_2^\infty$. It follows that $\Delta_n$ is periodic with period $M$. Let $\tau = \prod_0^{M-1} \Delta_n$; of course $\tau \ne 1$. There is some $L \in \mathbb{N}$ so that $\tau^L < \frac{1}{2}$ or $\tau^L > 2$.

Note that for any $n, m$, $|\Delta_n(\overline{\omega_0^n \dots \omega_m^n}) - (\overline{\omega_0^{n+1} \dots \omega_{3m}^{n+1}})| \le 4/(3m)$ (as in Lemma 6). In particular, for any $\epsilon > 0$, there is an $N$ such that $|\tau^L(\overline{\omega_0^0 \dots \omega_N^0}) - (\overline{\omega_0^{LM} \dots \omega_{3^{LM}N}^{LM}})| < \epsilon$ But then choosing sufficiently small $\epsilon$, we have $\overline{\omega_0^{LM} \dots \omega_{3^{LM}N}^{LM}} \notin [1, 2]$. This is a contradiction. □

The following stronger theorems shed some light on the structure of the tilings we will ultimately produce, but are not needed for Proposition 12 and so we omit the proofs.

**Theorem 10.** *In any orbit* $\{\mathtt{W}^n\} \subset \mathcal{L}^\infty_{12}$ *under* $\mathcal{R}^3$*, the words* $\mathtt{W}^n$ *have well-defined averages* $\overline{\mathtt{W}^n}$ *with maximum local imbalance* 8*. Moreover* $\overline{\mathtt{W}^n} = f(\overline{\mathtt{W}^{n-1}})$

Also, nicely: for an orbit $\mathcal{O} = \{\mathtt{W}_n\} \subset \mathcal{L}^\infty_{12}$ under $\mathcal{R}^3$, define the sequence $\delta = \delta(\mathcal{O}) \in \{1,2\}^{\mathbb{Z}}$ by $\delta_n = i$ if and only if $\mathtt{W}^n \in \mathcal{L}^\infty_i$. Then

**Theorem 11.** *For any orbit* $\{\mathtt{W}^n\} \subset \mathcal{L}^\infty$ *under* $\mathcal{R}$*,* $\delta(\mathcal{O})$ *is balanced with average* $\log_2 3$*. In particular,* $\delta(\mathcal{O}) = \omega(\log_2 3, \overline{\mathtt{W}^0})$*.*

# 4 The Tiles

Now consider the following set $\mathcal{T}$ twenty-six tiles, one for each letter of $\mathcal{A}$, drawn in the upper-half plane model of $\mathbb{H}^2$. All the tiles are bounded by horocycles above and below, and geodesics on the sides, and so are drawn as Euclidean rectangles in the upper half-plane model. Our tiles will have labeled edges and the matching rules will simply be the requirements that tiles sharing an edge have matching labels on that edge. (Such rules can be encoded geometrically, as bumps-and-nicks, if the reader chooses.) In Figure 3 we illustrate tiles corresponding to the example of Section 2.3 and Equation 2.

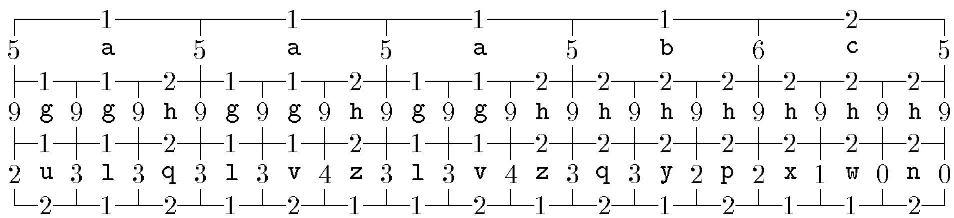



Figure 3: A portion of a tiling, corresponding to the right end of the example of Section 2.3 and Equation 2, by copies of tiles in $\mathcal{T}$.

Let $u, v, w$ be real parameters, $u > 0$, $1 < v < w < 3$. Define "templates"— tiles $T_{12}, T_3, T_4$ which we will soon decorate. Let $T_{12}$ have vertices with complex coordinates $wi, wi+u, wi+2u, wi+3u, 3i+3u, 3i$; let $T_3$ have vertices $vi, vi+u, wi+u, wi$; let $T_4$ have vertices $i, i+u, vi+u, vi$.

Note that the geometry of these tiles forces any tiling consisting of copies of $T_{12}, T_3, T_4$ to consist of successive horocyclic layers of $T_{12}$ tiles, then $T_3$ tiles, then $T_4$, then $T_{12}$ again, etc.

We next label these templates: The tiles $T_{\mathtt{a}}, \ldots, T_{\mathtt{f}}$ will be decorated copies of $T_{12}$; the tiles $T_{\mathtt{g}}, T_{\mathtt{h}}$ will be decorated copies of $T_3$; the tiles $T_{\mathtt{i}}, \ldots, T_{\mathtt{z}}$ will be decorated copies of $T_4$.

The labels $0, 1, \ldots, 9$ will be used for the "vertical" edges of the tiles; these labels correspond to the ten vertices of the graph describing $\mathcal{L}$ (Only the labels $0, \ldots, 4$ have numerical meaning): Each letter $\lambda \in \mathcal{A}$ runs from a vertex $i$ to a vertex $j$. Accordingly, the left edge $T_\lambda$ is decorated with the label $i$ and the right edge with the label $j$.

The labels 1, 2 will be used for the "horizontal" edges of the tiles: For $\lambda \in \{\mathtt{a}, \ldots, \mathtt{f}\}$ label the top edge $\overline{\lambda}$ and the bottom three edges with the labels $\overline{\mathtt{W}_1}, \overline{\mathtt{W}_2}, \overline{\mathtt{W}_3}$ where $\lambda \to \mathtt{W}_1\mathtt{W}_2\mathtt{W}_3$.

For $\lambda = \mathtt{g}, \mathtt{h}$, label both top and bottom edges of $T_\lambda$ with the label $\overline{\lambda}$.

For $\lambda = \mathtt{i}, \ldots \mathtt{z}$, $\lambda$ is produced by a unique letter, $\mu = \mathtt{g}, \mathtt{h}$. Label the top edge of $T_\lambda$ by $\overline{\mu}$. $\lambda$ may produce several letters $\nu$ but all have the same numerical value, with which we label the bottom edge.

All of this ensures that given any finite (infinite) horizontal strip $S = \{T_{\lambda_n}\}$ of tiles we may define a corresponding word $\mathtt{W}(S) = \{\lambda_n\} \in \mathcal{A}^*, (\mathcal{A}^{\mathbb{Z}})$. The construction assures that if $S$ satisfies the matching rules, $\mathtt{W} \in \mathcal{L}(\mathcal{L}^\infty)$, and conversely, for all $\mathtt{W} \in \mathcal{L}, (\mathcal{L}^\infty)$ there is a finite (infinite) strip $S$ with $\mathtt{W} = \mathtt{W}(S)$. Moreover, a strip $S_1$ and $S_2$ can be fitted together, $S_1$ "above" $S_2$ if and only if $\mathtt{W}(S_1)$ produces $\mathtt{W}(S_2)$. It follows that

**Proposition 12.** *The set $\mathcal{T} = \{T_\mathtt{a}, \ldots, T_\mathtt{z}\}$ is strongly aperiodic.*

**Proof** From the construction, any tiling by T corresponds to an orbit (perhaps with some choice in indexing) in $(\mathcal{A}, \mathcal{L}, \mathcal{R})$, and conversely, any orbit corresponds to a tiling. Consequently, as there exist orbits, there exists a tiling by $\mathcal{T}$.

Consider any tiling $X$ by tiles in T. No parabolic transformation $\rho$ can leave $X$ invariant since for any $\epsilon > 0$, there is a point $x$ with $0 < |x - \rho x| < \epsilon$. No hyperbolic transformation $\tau$ can leave $X$ invariant: Suppose not. Then $\tau$ must take horocyclic layers to horocyclic layers and leave some geodesic invariant. Indexing our horocyclic strips so that the 0th tile in each strip meets this geodesic, we thus produce a periodic orbit in $(\mathcal{A}, \mathcal{L}, \mathcal{R})$, which cannot occur. □

As a final note, one may reduce the number of tiles in several ways. For example, the tiles $T_\mathtt{b}, T_\mathtt{c}$ always occur together and can be joined to form one tile. Similarly one might join $T_\mathtt{e}$ and $T_\mathtt{f}$, and subsume $T_\mathtt{g}, T_\mathtt{h}$ into the other tiles. Less evidently, one may do away with $T_\mathtt{r}$— that is, there are orbits that make no use of the letters **r** (though every orbit does make use of each of the other letters).

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