Dodecafoam and substitution tilings

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Abstract

We discuss substitution tilings, certain hierarchical, highly structured tilings in the plane or more general geometric spaces. We use "Dodecafoam" as an artful, appealing example.

Substitution tilings are a class of highly structured hierarchical tilings in a geometric space; in essence, they are tilings generated by replacement rules and are geometric realizations of regular languages (a general, theoretical discussion of this can be found in [3]). We will begin with a quick formal discussion and examples, then give general algorithms for producing these tilings; finally we will discuss the dodecafoam example.

Formalities

We will assume we are working in $\mathbb{E}^n$, n-dimensional Euclidean space (though we can generalize all these definitions); $\mathcal{G} = Isom(\mathbb{E}^n)$ is the set of isometries acting on $\mathbb{E}^n$. A <u>*tile*</u> is some compact set of points that is the closure of its interior.[1] Given a set T of tiles, a <u>*configuration of tiles in T with support $U \subset \mathbb{E}^n$*</u> is a collection $\tau = \{gA\}$ of tiles in $\mathcal{G}T$ such that (1) for all $gA, hB \in \tau$, gA and hB have disjoint interiors and (2) $U = \bigcup_{gA\in\tau} gA$. A <u>*tiling*</u> is a configuration with support $\mathbb{E}^n$; a <u>*species*</u> is any collection of tilings.

A <u>*substitution*</u> is defined by three components: a set of tiles T, an expansion constant s acting on $\mathbb{E}^n$ by $s(\mathbf{x}) = s\mathbf{x}$, and a map σ taking each tile $A \in T$ to a configuration of tiles in T with support sA. Now in particular, for each $A \in T$, $\sigma A = \{g_i B_i\}$ for some collection of tiles in $\mathcal{G}T$.

So for example, in figure 1, on the upper left, we see that a pentagon can be divided into three kinds of tiles in $\mathbb{E}^2$: a pentagon, p_0, and two isoceles triangles h_0 and w_0. If we take as our inflation constant the famous golden ratio, $s = \phi = \frac{1}{2}(1+\sqrt{5})$, and define $\mathsf{p}_1 = \phi\mathsf{p}_0$, $\mathsf{h}_1 = \phi\mathsf{h}_1$ we can then let our substitutions

[1] That is, we rule out certain crazy pathologies.

be: $\sigma(\mathsf{p}_0) = \{\mathsf{p}_1\}$, and $\sigma(\mathsf{h}_0) = \{\mathsf{h}_1\}$; we let $\sigma(\mathsf{p}_1)$ be a configuration with five congruent copies of h_0, five of w_0 and one of p_0, as pictured (for brevity, we will rely on the figure to specify the eleven isometries used to get these congruent copies; note that these isometries are critical to the construction). Similarly, the configurations $\sigma(\mathsf{h}_1)$, $\sigma(\mathsf{w}_0)$ are illustrated.

For any $g \in \mathcal{G}$, we define $\sigma(gA)$ in the obvious way; formally, this works out to be: $\sigma(gA) = sgs^{-1}\sigma(A)$. For any configuration τ we can define $\sigma(\tau) = \bigcup_{gA\in\tau} \sigma(gA)$

We can thus iterate this substitution on tiles and define $\sigma^2(A) = \sigma(\sigma(A))$, and so on. Any configuration of the form $g\sigma^n(A)$, $g \in \mathcal{G}, n \in \mathbb{N}, A \in T$, we will call an *n-level supertile*.

And now we can define:

Fixing our tiles T, expansion constant s and rules σ, a tiling τ is a *substitution tiling* if and only if for every bounded configuration $\tau' \subset \tau$, there exist $g \in \mathcal{G}$, $A \in T$ and $n \in \mathbb{N}$ with $\tau' \subset g\sigma^n(A)$. That is, a tiling is a substitution tiling if in any bounded region it "looks" like part of an n-level supertile. Let $\Sigma(T, \sigma)$ be the species of all such substitution tilings based on T, s and σ. It is non-trivial to show that $\Sigma(T, \sigma)$ is non-empty, or to describe its structure in any detail; this is seen in [3], [5], [8] and elsewhere.

So a portion of a substitution tiling derived from the pentagonal substitution given above is illustrated at right in figure 1

A few facts may be worth noting: A tiling τ is said to be *periodic* if some fixed-point free isometry g satisfies $g\tau = \tau$ (thus the familiar tiling of $\mathbb{E}^2$ by a lattice of squares is periodic) and *non-periodic* otherwise.[2]

In some generic, but ill-defined sense, most substitution tilings consist of only non-periodic tilings. For any particular example this is shown using the following idea:

A subsitution species $\Sigma(T, \sigma)$ has *unique decomposition* if and only if the map $\sigma : \Sigma(T, \sigma) \to \Sigma(T, \sigma)$ is

[2] It is very important not to confuse this with *aperiodicity*. A set T of tiles is *aperiodic* if every possible tiling by tiles in T is non-periodic. Subsitution species indirectly provide one of the main methods of producing aperiodic sets of tiles, though. See [4] or [8] for detailed discussion.

one-to-one. Effectively this means that any tiling in the species has a uniquely defined hierarchy of supertiles; that is, that each tile in any τ in the species is in a unique nested collection supertiles, one of each level.

It is a well-known theorem [8] that if a substitution species has unique decomposition then it is non-periodic;[3] and as a practical point, this is usually how non-periodicity is shown (for a very typical proof, see [6] or [7]). The example of figure 1 has unique decomposition and is thus non-periodic.

Interestingly, if a substitution species *does* have unique decomposition, then it contains uncountably many tilings.

Finally, though substitution tilings are not usually periodic, if certain pathological cases are ruled out, they are *repetitive*: that is, given any species $\Sigma(T, \sigma)$ satisfying very mild conditions, given any $\tau \in \Sigma(T, \sigma)$ and any bounded configuration $\tau' \in \tau$, there is an $R > 0$ such that in *any* configuration τ_0 of radius R, in *any* tiling in $\Sigma(T, \sigma)$, there is an image $g\tau' \subset \tau_0$. So in a very strong sense, tilings in a given substitution species all look pretty much the same, and look the same everywhere. This is certainly believable after looking at a few examples!

In figure 2 we see several substitution tilings in the Euclidean plane. The variety of structure is striking. And we are not really limited by the setting set forth above; in figure 3 we have an example of a substitution tiling with quite a different kind of structure. Here the replacement transformations are projective, not Euclidean. Even in the Euclidean case there are many open questions: for example, even in that restricted setting it is still not known whether there are uncountably many combinatorially distinct substitution tilings.

Algorithms

The above discussion points to a pretty concrete way to generate substitution tilings. We will write our code in *Mathematica*. The following framework is suitable for a range of applications, from symbolic

[3] The converse is only partially understood; *cf* [11]

dynamics to dodecafoam. We use the following function:[4]

```
SubIt[object0_,depth_,subrule_,getmotif_,showmotif_]:=
      showmotif[getmotif /@
            Nest[(Join @@ subrule /@ #)&, {object0},depth]]
```

The substitution acts on `objects`, which can be any data structure we please. For the tiling examples, these will be of the form {`position, type`}.

We begin with an initial object, `object0` and iterate `depth` times. The rules for the substitution are given by a function `subrule` that takes an `object` as input and returns a list of `objects`. I prefer to divide the final display into two functions; the first, `getmotif` takes as input an `object` and puts this into some intermediate form. The second, `showmotif` takes a list of outputs from `getmotif` and produces the final product.

Effectively, then, given the input just described, `SubIt` does the following: at each step, `SubIt` keeps a list of objects; the function `subrule` is applied to each object in the list, each producing a list of objects; these lists are then combined, and the process is iterated `depth` times. Then `getmotif` is applied to each object in this list and `showmotif` is applied to this as a whole. Let us give two examples.

A symbolic dynamics example: Here is a simple string replacement operation; we have an alphabet of two letters, 0 and 1; our "tilings" are strings of 0's and 1's. At each step of the substitution we replace any occurence of 0 with 01 and any occurence of 1 with 10. Thus, $0 \to 01 \to 0110 \to 01101001 \to 0110100110010110 \to \ldots$. The commands we define to use in `SubIt` are:

[4] In *Mathematica*, `f @@ {a, b, ... }` yields `f[a, b, ... ]`; `f /@ {a, b, ... }` yields `{f[a], f[b], ... }`. `Join` joins lists into one list; an expression including `#` and ending in `&` is a pure function, hence `(#^2 +2)&[3]` yields `11`. `Nest[f, a, n]` gives $\underbrace{\mathtt{f}[\mathtt{f}[\ldots\mathtt{f}[}_{n}\mathtt{a}]]]$

```
letterule["0"]={"0","1"};

letterule["1"]={"1","0"};


lettermotif[letter_]:=letter


showletter[letterlist_]:=StringJoin @@ letterlist
```

Evaluating

```
SubIt["0", 4, letterule, lettermotif, showletter]
```

we get as output: *0110100110010110*

A tiling example: For a tiling example in $\mathbb{E}^n$, things are somewhat more complex; We will just sketch a few of the basic points:

Suppose our tiles are to be called $\{\mathtt{X},\mathtt{Y}\}$, our inflation constant is s, and we have that $\sigma(\mathtt{X}) = \{\mathtt{aX},\mathtt{bY}\}$, $\sigma(\mathtt{X}) = \{\mathtt{cX},\mathtt{dY}\}$ where $\mathtt{a},\mathtt{b},\mathtt{c},\mathtt{d} \in \mathcal{G}$. We will represent our points in $\mathbb{E}^n$ as $(n+1)$-dimensional homogeneous column vectors; $\mathcal{G}$ will consist of $(n+1)\times(n+1)$ matrices.

Then our `objects` will be of the form {`position, type`}; where `position` is an $(n+1)\times(n+1)$ matrix. Let `S` be the $(n+1)\times(n+1)$ matrix decribing inflation by s, let `SS` be $\mathtt{S}^{-1}$, and let `Ident` be the $(n+1)\times(n+1)$ identity matrix. We can define:

```
tilerule[{pos_,X}]:={{S . pos . SS . a, X}, {S . pos . SS . b, Y}};

tilerule[{pos_,Y}]:={{S . pos . SS . c, X}, {S . pos . SS . d, Y}};
```

`tilemotif[{pos_,type_}]` then returns a list of appropriate graphics primitives that depend of the type of tile and its position, and `showtiles[graphicslist_]` takes a list of lists of graphics primitives, combines these into one list and displays the result, perhaps after additional formatting. We thus evaluate something like:

```
SubIt[{Ident,X}, 6, tilerule, tilemotif, showtile]
```

Dodecafoam

We now discuss a favorite example, Dodecafoam. The author has known the example partially since about 1988 and completely since 1993. This particular substitution tiling is closely related to a three dimensional analogue of the Penrose tiles; one version is described in [10]; these tiles in turn appear to be related to a range of physically occuring "quasicrystals" [9]. Other related substitution species are given in [1].

I was led to the construction in part by thinking about generalizations of the famous Koch snowflake. David's fractal stellated dodecahedron [2], was thought of along these lines and can be found within this construction.

And so let us turn to the construction ... (The precise details of the construction are relegated the appendix.)

We begin with a regular dodecahedron D_3 with edges of length ϕ^3 where $\phi = \frac{1+\sqrt{5}}{2}$. We "facet" D_3; that is, we divide D_3 using twelve planes cutting through its interior as pictured in figure 4. This divides the dodecahedron into several different types of cells.

Before describing the cells, let us define triangles $\mathsf{h}_i = \phi^i \mathsf{h}_0$ and $\mathsf{w}_i = \phi^i \mathsf{w}_0$ and pentagons $\mathsf{p}_i = \phi^i \mathsf{p}_0$. Note that the planes used to facet D_3 cut each other into one copy of p_0, and five copies each of h_0, h_1, h_2, and w_0 and ten of w_1 (figure 4). The cells in the faceted D_3 will have faces that are of these types.

In the very center of our faceted D_3 lies a small dodecahedron of with faces all congruent to p_0 (figure 5, a). We'll call this dodecahedron D_0; note that $\phi^3 \mathsf{D}_0 = \mathsf{D}_3$.

On each face of D_0 we see a pentagonal pyramid, or "small hat" H_0; these have one face congruent to p_0 and five faces congruent to h_0. D_0 and the twelve small hats together form a small stellated dodecahedron (figure 5, b & c).

Between neighboring hats, we next see a total of thirty "small wedges" W_0. These wedges are tetrahedra with two copies each of h_0 and w_0 as faces. All together, the dodecahedron D_0, the twelve small hats H_0, and the thirty small wedges form a great dodecahedron (figure 5, d & e).

Next we see twenty "spikes" S_0; these spikes fit into the triangular depressions of the great dodecadron to form a great stellated dodecahedron. These spikes are trigonal dipyramids— that is, two triangular pyramids stuck together. The inner pyramids have faces congruent to w_0; the outer pyramids have faces congruent to h_2. The convex hull of this great stellated dodecahedron is our original dodecahedron D; that is, the vertices of D coincide with the tips of the spikes (figure 5, f & g).

We now fill in the remaining parts of D. Between the spikes we fill in thirty strangely shaped "chocks" C_0; the faces of each chock are two copies of h_2 and six copies of w_1 (figure 5, h). As it turns out, we can replace each chock with a copy of the dodecahedron D_0, four small hats and a wedge (figure 6, a); and so we will eventually dispose of the chock entirely.

Finally, we can fill in the star shaped holes that remain (figure 5, i) with twelve "large hats" $\mathsf{H}_1 := \phi \mathsf{H}_0$ and sixty "large wedges" $\mathsf{W}_1 := \phi \mathsf{W}_0$.

Thus the twelve planes cutting through the interior of D divide D into 165 cells— one small dodecahedron, twelve small hats, thirty small wedges, twenty spikes, thirty chocks, sixty large wedges and twelve large hats. Or, replacing chocks with other pieces, the large dodecahedron is divided into 345 cells! But this is not nearly as bad as it sounds— there is a tremendous amount of symmetry to be taken advantage of.

These cells— dodecahedra, hats, wedges, spikes and chocks— will be the basic pieces of the substitution tiling. It now remains to decide on the most "natural" way to subdivide these cells. The basic idea is to use the faceting of D_3 to find "natural" subdivisions— that is, within the 165 cells described above, look for clusters that are enlarged versions of our basic cells.

It is unclear to me, frankly, whether the particular choices made are art or science. But this hueristic idea works again and again for similar constructions based on other polytopes. At any rate, centered within the faceted D_3 we can find a copy of $\mathsf{H}_2 := \phi^2 \mathsf{H}_0$ (figure 6,b); a copy of $\mathsf{W}_3 := \phi^3 \mathsf{W}_0$; and a copy of $\mathsf{S}_1 := \phi \mathsf{S}_0$.

Thus, after two inflations by ϕ, we can replace our small hats H_0; after three inflations we can replace our small wedges; after one inflation, our spikes, and after three inflations we can replace our small dodecahedra. We thus set our inflation factor to be ϕ and define the substitution as:

$\mathsf{H}_0 \to \mathsf{H}_1 \to \mathsf{H}_2$ which is then replaced with a collection of cells.

$\mathsf{W}_0 \to \mathsf{W}_1 \to \mathsf{W}_2 \to \mathsf{W}_3$ which is then replaced with a collection of cells.

$\mathsf{S}_0 \to \mathsf{S}_1$ which is then replaced with a collection of cells.

$\mathsf{D}_0 \to \mathsf{D}_1 := \phi D_0 \to \mathsf{D}_2 := \phi^2 \mathsf{D}_0 \to \mathsf{D}_3$ which is then replaced with a collection of cells.

We then iterate the substitution, using something like `SubIt`.

In fact, though, the nice images of dodecafoam are not quite strictly of the substitution tiling described above. There is really only one change— we never substitute in for dodecahedra. The last chain of substitutions is actually:

$\mathsf{D}_0 \to \mathsf{D}_1 \to \mathsf{D}_2 \to \ldots \to \mathsf{D}_n \to \ldots$

And we draw only the dodecahedral cells; all the other cells are simply left out at the final step. So, in figure 7, the hat, wedge, spike and chock are illustrated after a few iterations of the substitution.

I had been making hand-drawn (!) pictures and paper models of dodecafoam for a few years when I finally had an opportunity to make some high quality computer renderings. The transformation rules, given as 4×4 matrices acting on homogeneous coordinates, were first assembled from basic operations, as described below, in *Mathematica*. Dan Krech, then an apprentice at the Geometry Center at the University of Minnesota, wrote a program to iterate the substitution— that is, to repeatedly multiply all these transforms together (this is faster than using the `SubIt` routine described above). The images were then assembled in *Geomview* (available from the Geometry Center), which took as input simply a list of the transforms acting on the motif, a dodecahedron. Since the transforms were given separately from the motif, any *Geomview* object could be used as the motif; some amusing images were made with human heads in place of the dodecahedra. Lights, colors and materials were chosen in *Geomview*. Finally, for the four images in this article, the files were passed onto *Renderman* for final rendering.

The images in figures 8, 10, 11, and 9, differ in the choice of cells at the begining of the process; thus we fill a stellated dodecahedron, great dodecahedron, great stellated dodecahedron and dodecahedron are filled with dodecafoam; these solids are nested, exactly as in figure 5.

Appendix: Details of the construction

We use the following transformations as building blocks:

First define ρ_x, ρ_y, ρ_z as the rotations about the x, y and z axes by 180 degrees, and r_x, r_y, r_z as reflections along the x, y and z axes. So, for example $\rho_x(x,y,z) = (x,-y,-z)$ and $r_x(x,y,z) = (-x,y,z)$. Define ρ_3 as the clockwise rotation by 120 degrees about the vector $(1,1,1)$; thus $\rho_3(x,y,z) = (y,z,x)$. Define ρ_5 as the clockwise rotation by 72 degrees about the vector $(1,0,\phi)$; thus, considering a row vector $\mathbf{x} \in \mathbb{E}^3$, $\rho_5(\mathbf{x})$ is given by

$$\rho_5(\mathbf{x}) = \mathbf{x} \cdot \frac{1}{2} \begin{pmatrix} 1 & -\phi & \phi^{-1} \\ \phi & \phi^{-1} & -1 \\ \phi^{-1} & 1 & \phi \end{pmatrix}$$

Note that these rotations, individually, generate cyclic groups of order 2, 3 or 5. So for example: $<\rho_3>= \rho_3^{\{0,1,2\}} = \{\rho_3^0, \rho_3^1, \rho_3^2\}$. For further convenience we write: $< r|s|..|t >$ for $< r >< s > \ldots < t >= \{r^i s^j \ldots t^k\}$. We have:

The vertices of D_0 are: $<\rho_3> (\pm 1, 0, \pm\phi^{-2})$ and $(\pm\phi^{-1}, \pm\phi^{-1}, \pm\phi^{-1})$

The vertices of the small stellated dodecahedron are: $<\rho_3> (\pm 1, 0, \pm\phi)$.

The vertices of D_3 are of course $<\rho_3> (\pm\phi^3, 0, \pm\phi), (\pm\phi^2, \pm\phi^2, \pm\phi^2)$

Choosing particular cells to represent $\mathsf{H}_0, \mathsf{W}_0, \mathsf{S}_0$, and C_0, we have that the vertices of H_0 are $< \rho_5 >$ $(\phi^{-1}, \phi^{-1}, \phi^{-1})$ and $(1, 0, \phi)$.

The vertices of W_0 are $(\pm 1, 0, \phi)$, $(0, \pm\phi^{-2}, 1)$.

The vertices of S_0 are $<\rho_3> (1, 0, \phi), (\phi^2, \phi^2, \phi^2)$ and $(\phi^{-1}, \phi^{-1}, \phi^{-1})$.

The vertices of C_0 are $(\pm 1, 0, \phi), (0, \pm\phi, \phi^3), (\pm 1, 0, \phi + 2)$.

For any cell X_0, we define $X_i := \phi^i X_0$.

We have the following substitution rules:

$\mathsf{H}_0 \to \mathsf{H}_1$

$\mathsf{H}_1 \to \{\mathsf{D}_0,\ \mathsf{H}_0,\ <\rho_5>\rho_x \mathsf{H}_0,\ <\rho_5>\rho_3^2\rho_x \mathsf{W}_0\} + (1, 0, \phi)$

$\mathsf{W}_0 \to \mathsf{W}_1 \to \mathsf{W}_2$

$\mathsf{W}_2 \to \{\mathsf{D}_0,\ \rho_3^2 <\rho_x|\rho_z> \mathsf{H}_0,\ \rho_3 <\rho_x> \rho_z \mathsf{H}_0,\ <\rho_z>\mathsf{H}_0,$

$\mathsf{W}_0,\ <\rho_z>\rho_5<\rho_3>\rho_5^2\mathsf{W}_0,\ <r_x|\rho_z>\rho_5^2\rho_3\mathsf{W}_0,\ <\rho_z>\rho_5\mathsf{S}_0\} + (0, 0, 2\phi)$

$\mathsf{S}_0 \to \{\mathsf{D}_0,\ \mathsf{S}_0,\ <\rho_3|\rho_x> \mathsf{H}_0,\ <\rho_3>\rho_5^{\{1,4\}}\mathsf{W}_0\} + (1, 1, 1)$

$\mathsf{D}_0 \to \mathsf{D}_1 \to \mathsf{D}_2$

$\mathsf{D}_2 \to \{\ \mathsf{D}_0,$

$<\rho_3|\rho_x|\rho_z> \mathsf{H}_0,$ the small stellated dodecahedron

$<\rho_5|\rho_3|\rho_x> \mathsf{W}_0$ the great dodecahedron

$<\rho_5|\rho_x|\rho_z> \mathsf{S}_0,$ great stellated dodecahedron

$<\rho_5|\rho_3|\rho_x> \mathsf{C}_0,$ filling in the outside of D_3

$<\rho_5|\rho_3|\rho_x|\rho_z> (\rho_x^1\mathsf{W}_1 + (\phi, 0, \phi^2)),$

$<\rho_3|\rho_x|\rho_z> (\rho_y^1\mathsf{H}_1 + (\phi, 0, \phi^2))\quad \}$

where:

$\mathsf{C}_0 = -\{\mathsf{D}_0, \mathsf{W}_0, <\rho_z>\mathsf{H}_0, \rho_3<\rho_x>\rho_z\mathsf{H}_0\} + (0, 0, 2\phi)$

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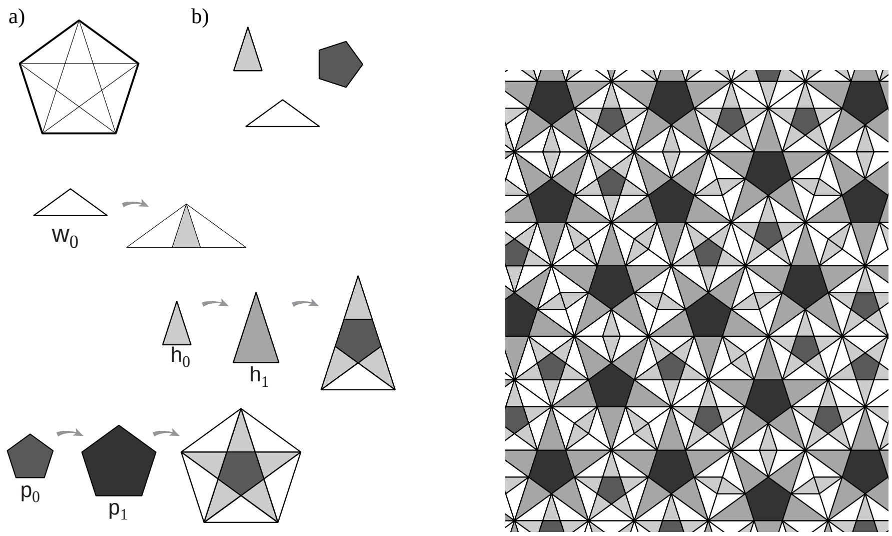

Figure 1: A pentagonal substitution tiling

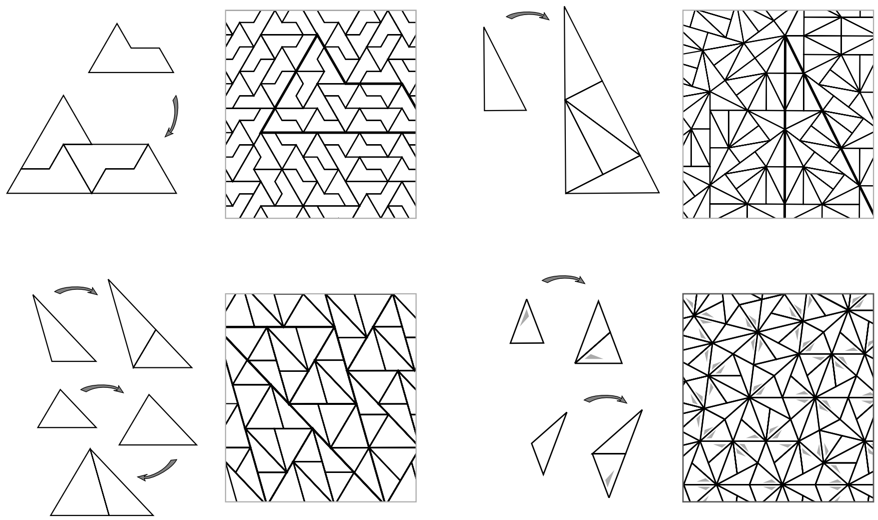

Figure 2: Several substitution tilings

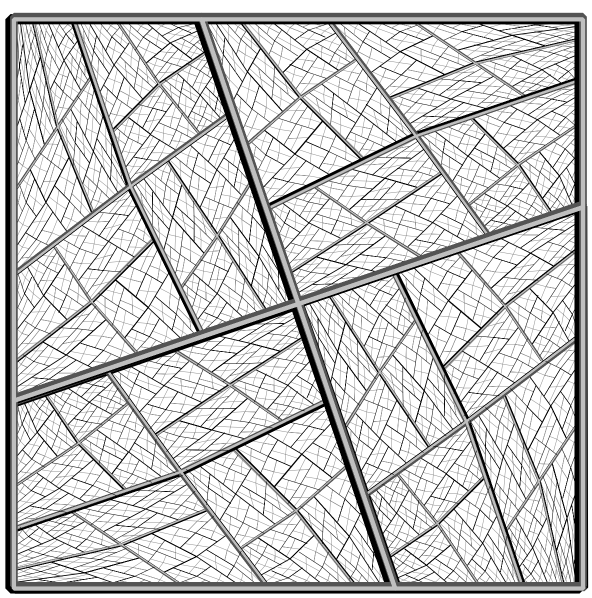

Figure 3: A projective substitution tiling

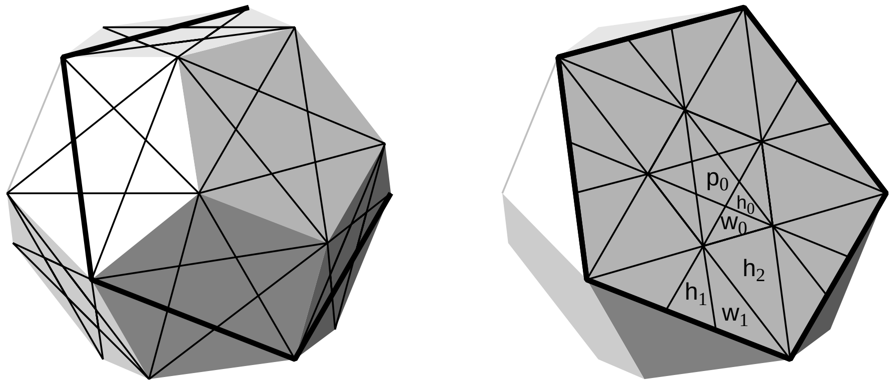

Figure 4: Faceting D_3

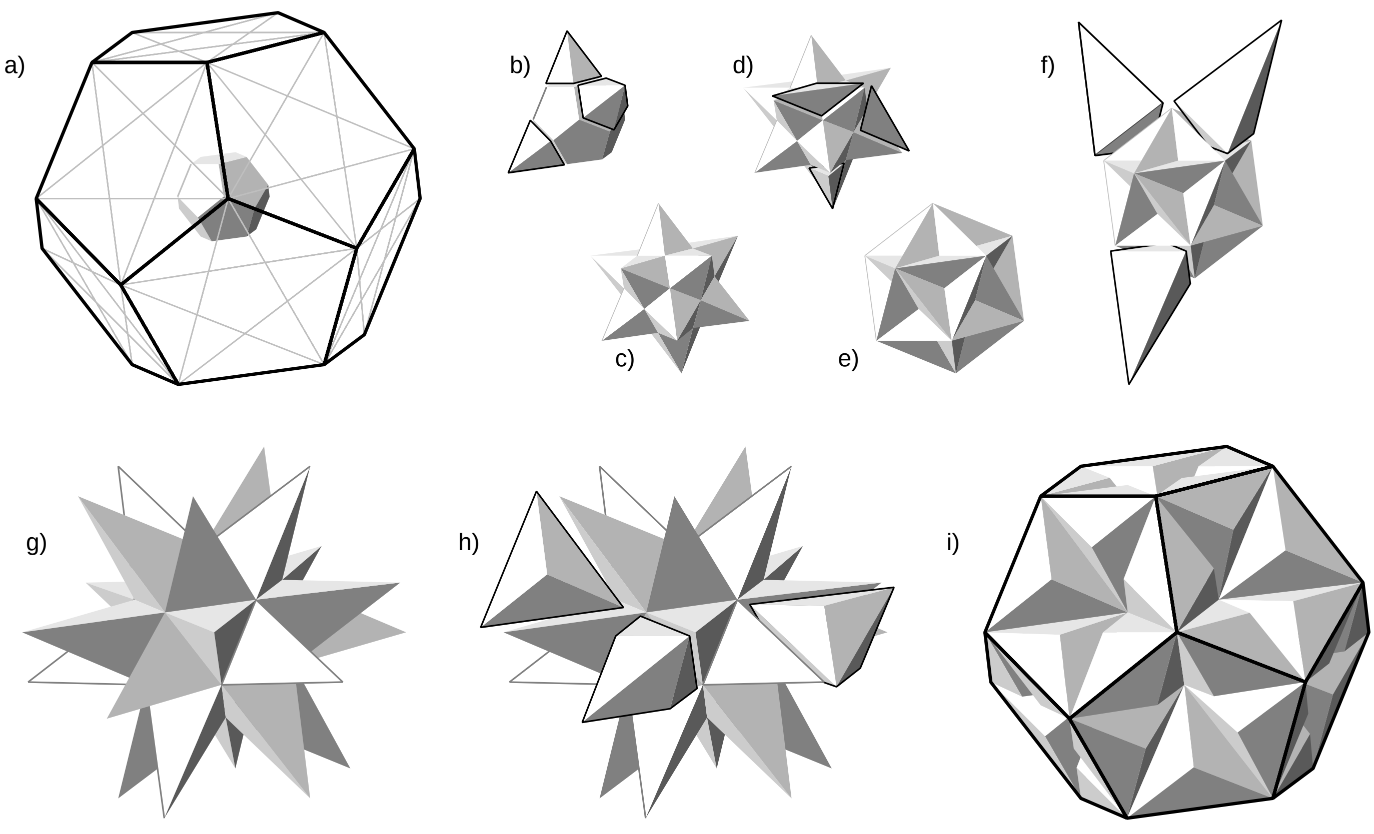

Figure 5: Examining the faceted D_3

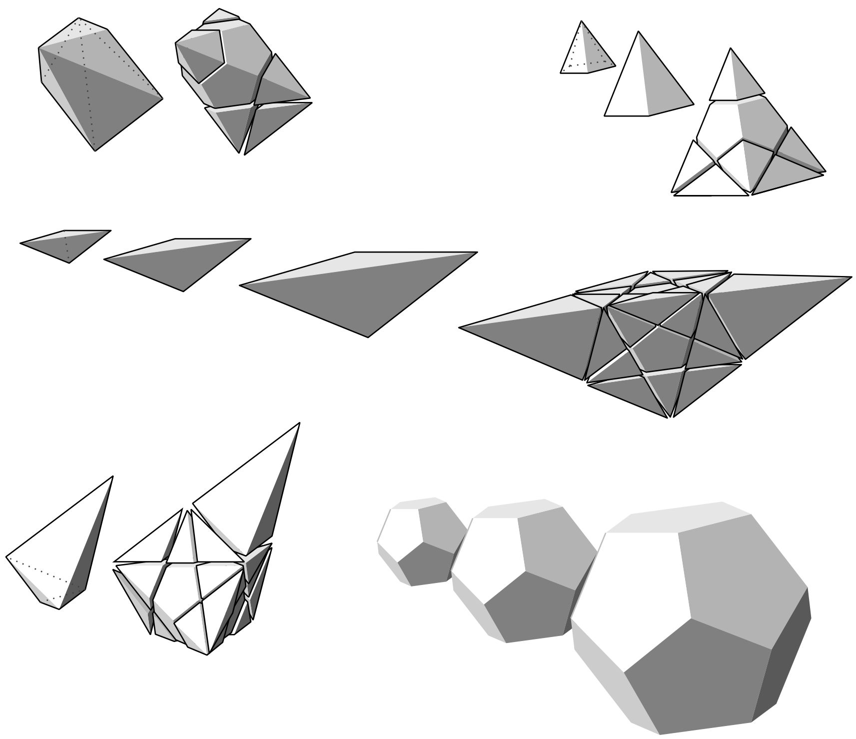

Figure 6: Substitutions

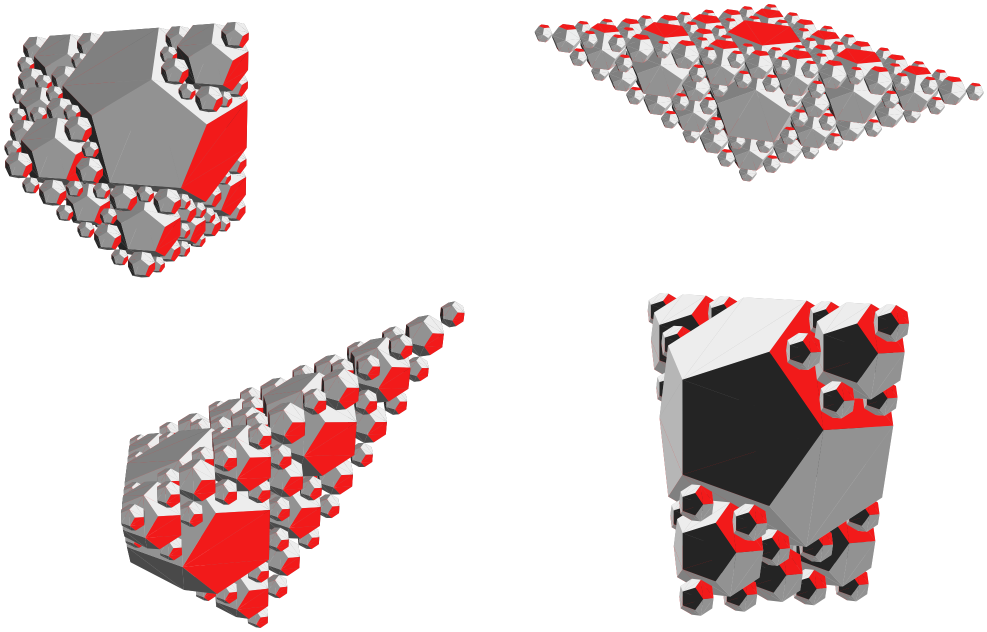

Figure 7: Filling the cells with dodecahedra

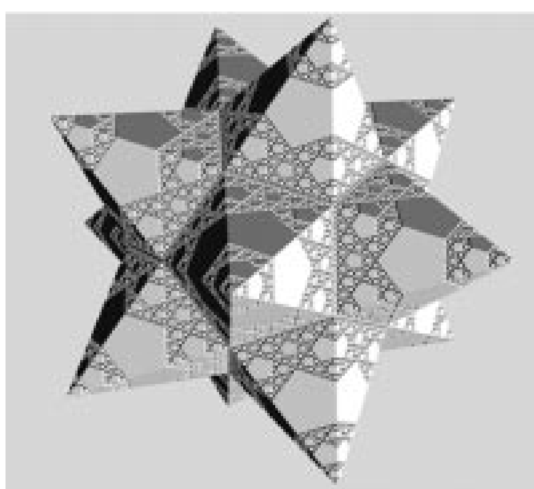

Figure 8: Dodecafoam filling the stellated dodecahedron

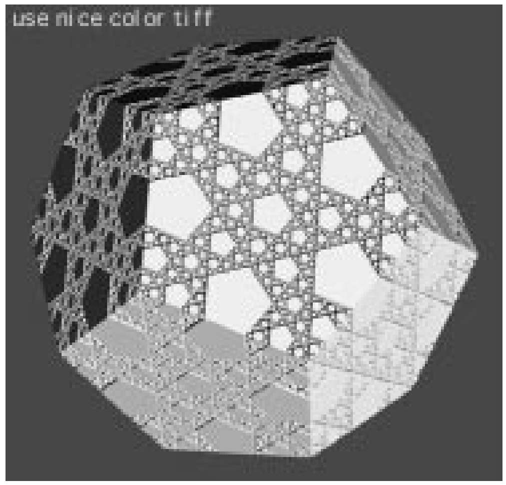

Figure 9: Dodecafoam filling a dodecahedron

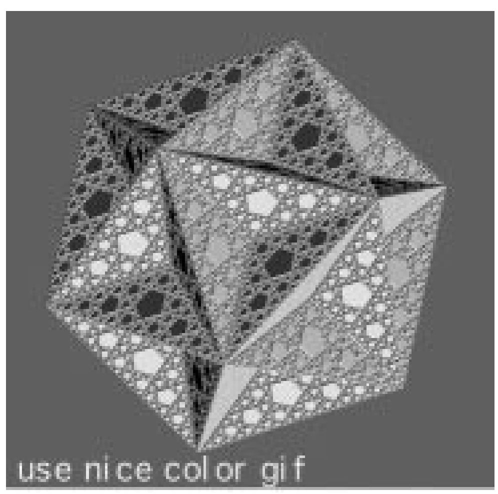

Figure 10: Dodecafoam filling the great dodecahedron

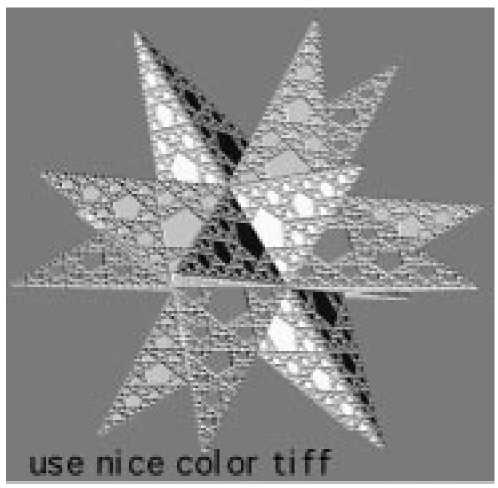

Figure 11: Dodecafoam filling the great stellated dodecahedron