Piecewise-Smooth Surfaces as the Union of Geodesic Disks

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**Abstract:** We consider a question of G. Fejes Tóth, whether the boundary of a convex body can be the union of three geodesic disks with disjoint interior.

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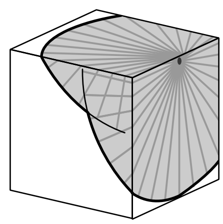

Figure 1: A geodesic disk

We are motivated by the following definition and question of G. Fejes Tóth [1]:

A **geodesic disk** (Figure 1) is a piecewise smooth disk $D$ embedded in $\mathbb{R}^3$ such that there exist a center $c \in D$, and radius $r \in \mathbb{R}$, $r > 0$, with
(a) for all $x \in \partial D$, $d(x, c) = r$ and
(b) for all $x \in int(D)$, $d(x, c) < r$,
where $d$ is the extrinsic metric on $D$ (i.e. distance measured along shortest geodesics on $D$) and where a surface, possibly with boundary, is **piecewise smooth** if it is the union of $C^2$ surfaces, with disjoint interiors and piecewise $C^2$ boundaries.

**Question:** *Does there exist a convex body in $\mathbb{R}^3$ such that $\partial B$ is the union of three geodesic disks with disjoint interiors.*

We will show, as a corollary to a more general theorem, that the answer is no, at least under the reasonable assumption that the convex body is piecewise smooth.

Let $\mathcal{F}$ be a compact piecewise smooth surface, possibly with boundary, embedded in $\mathbb{R}^3$. For any point $x$ on $\mathcal{F}$ we define the **local excess curvature** $\kappa_{\mathcal{F}}(x)$ of $\mathcal{F}$ at $x$ as follows: Since $\mathcal{F}$ is piecewise smooth, for $x \in \mathcal{F}$, $x$ is incident to some finite collection $\{\mathcal{F}_i\}$ of smooth subsets of $\mathcal{F}$. Each $T_x\mathcal{F}_i$ is a sector of a plane centered at $x$ with angle measure $a_i$. Then for $x \in \partial\mathcal{F}$, we define $\kappa_{\mathcal{F}}(x)$ as:

$$\kappa_{\mathcal{F}}(x) := (\Sigma a_i) - \pi$$

And for $x \in$ *int*$(\mathcal{F})$, we define $\kappa_{\mathcal{F}}(x)$ as:

$$\kappa_{\mathcal{F}}(x) := (\Sigma a_i) - 2\pi$$

If $x \notin \mathcal{F}$, define $\kappa_{\mathcal{F}}(x) = 0$.

Let $\mathcal{F}$ be a compact piecewise smooth surface, embedded in $\mathbb{R}^3$ that is the finite union of geodesic disks $D_i$ with disjoint interiors. Consider the set $\mathcal{V}$ of vertices of the graph $\Gamma = \mathcal{F} \setminus \cup\, int(D_i)$ and define

$$\kappa(\mathcal{F}) := \sum_{x \in \mathcal{V}} \kappa_{\mathcal{F}}(x)$$

**Theorem 1** *Suppose a compact piecewise smooth surface $\mathcal{F}$, possibly with boundary, embedded in $\mathbb{R}^3$, is the union of $n$ geodesic disks with disjoint interiors. Then $\kappa(\mathcal{F}) \geq 2\pi(n - \chi(\mathcal{F}))$.*

This immediately gives:

**Corollary 2** *Let $B$ be a convex body in $\mathbb{R}^3$ with piecewise smooth boundary. Then $\partial B$ cannot be the union of more than two geodesic disks with disjoint interiors.*

This follows immediately from the observation that if $B$ is convex, then $\kappa_{\partial B}(x) \leq 0$ for all $x \in \partial B$ and that $\partial B$ must be a topological sphere. Consequently if $\partial B$ is the union of $n$ geodesic disks with disjoint interiors, $0 \geq 2\pi(n-2)$ and so $n \leq 2$.

**Corollary 3** *Let $\mathcal{F}$ be a smooth surface, possibly with boundary, embedded in $\mathbb{R}^3$ such that $\mathcal{F}$ is the union of $n$ geodesic disks with disjoint interiors. Then either $\mathcal{F}$ is a topological sphere and $n = 2$, or $\mathcal{F}$ is itself a geodesic disk and $n = 1$.*

This follows immediately from the observation that if $\mathcal{F}$ is smooth, $\kappa(x) = 0$ for all $x \in \mathcal{F}$. Hence $n = \chi(\mathcal{F})$. Finally, $\chi \leq 2$ and $n > 0$, and so $\chi = n = 1, 2$.

The theorem follows quickly from the following lemma:

**Lemma 4** *Let $D$ be a geodesic disk. Then for all $x \in \partial D$, $\kappa_D(x) \geq 0$*

We will show that the theorem follows from the lemma, and then prove the lemma.

**Proof of Theorem 1:** Let $\mathcal{F}$ be a compact piecewise smooth surface, possibly with boundary, embedded in $\mathbb{R}^3$, such that $\mathcal{F}$ is the union of $n$ geodesic disks $D_i$. Let $\Gamma = \mathcal{F} \setminus \cup D_i$, the finite graph of the boundaries between these disks. The complement of $\Gamma$ has $n$ components. Let $\mathcal{V}$ be the set of vertices of $\Gamma$; let $v$, $e$ be the number of vertices, edges of $\Gamma$. For every $x \in \mathcal{V}$, let $q_x$ be the number of edges of $\Gamma$ meeting $x$. Of course $q_x \geq 2$ and so the lemma gives, for all $x \in \mathcal{V}$,

$$\kappa(x) > q_x\pi - 2\pi$$

( Note that this holds whether $x$ is on the boundary or in the interior of $\mathcal{F}$.) Thus:

$$\begin{aligned}
\sum_{x \in V} \kappa_{\mathcal{F}}(x) &\geq \sum_{x \in V} (q_x\pi - 2\pi) \\
&= 2\pi(e - v) \\
&= 2\pi(n - \chi(\mathcal{F}))
\end{aligned}$$

□

We now turn to the proof of the lemma.

**Proof of Lemma 4:** Let $D$ be a geodesic disk with center $c$ and radius $r$. Since $D$ is piecewise smooth, we can linearly approximate to arbitrary degree a sufficiently small neighborhood of $x \in D$. In particular, in a sufficiently small neighborhood of $x$, $D$ is arbitrarily close to the union of flat sectors with apex at $x$, and geodesics are arbitrarily close to straight lines.

Let $x \in \partial D$ and let $\gamma : [0, r] \to D$ be the path of shortest length from $c$ to $x$. Let $\lambda$ be a portion of the boundary of $D$ on either side of $x$. Then we claim that the angle between $\lambda$ and $\gamma$ in $D$ at $x$ must be at least $\pi/2$.

For if not, let $x_1$ be a point very near to $x$ on $\lambda$. Take a geodesic perpendicular to $\partial D$, meeting $x_1$, and passing through $\gamma$, at some point $a$

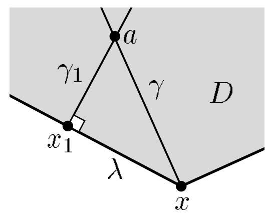


Figure 2: A contradiction arises if the angle between $\gamma$ and $\lambda$ is not at least $\pi/2$

(Figure 2). Now for $x_1$ sufficiently close to $x$, $d(a, x_1) < d(a, x)$ whence:

$$\begin{aligned} d(c, x_1) &\leq d(c, a) + d(a, x_1) \\ &< d(c, a) + d(a, x) \\ &= d(c, x) \end{aligned}$$

Thus, $d(c, x) \neq d(c, x_1)$, which contradicts the assumption that $x, x_1 \in \partial D$.

It immediately follows that $\kappa_D(x) \geq 0$, since the total angle at $x$ in $D$ must be at least $\pi$.

$\square$

**Reference**

[1] G. Fejes Tóth, Problem 12b, Discrete Geometry Tagungsbericht 20/1997, Math. Forschungsinstititut Oberwolfach.

[2] H. Busemann, The geometry of geodesics, Academic Press, New York (1955).

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