

# Regular Production Systems and Triangle tilings

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## Abstract

We discuss *regular production systems* as a tool for analyzing tilings in general. As an application we give necessary and sufficient conditions for a generic triangle to admit a tiling of  $\mathbb{H}^2$  and show that almost every triangle that admits a tiling is “weakly aperiodic.” We pause for a variety of other applications, such as non-quasi-isometric maps between regular tilings, aperiodic Archimedean tilings, growth, and decidability.

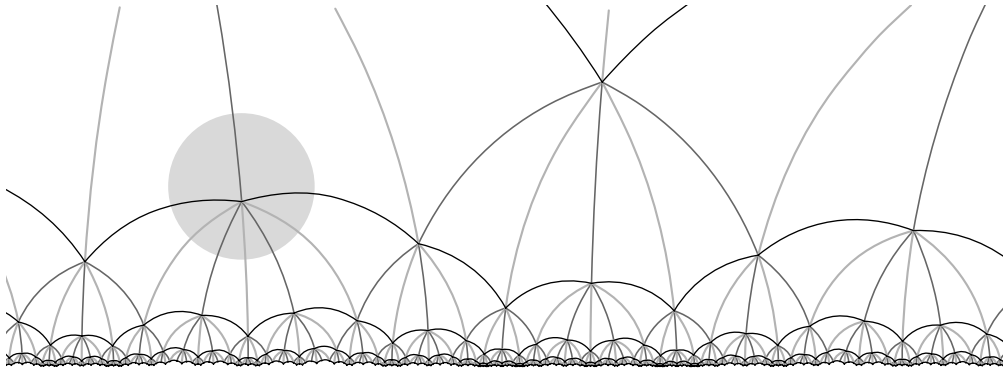


Figure 1: A tiling of  $\mathbb{H}^2$  by copies of a triangle whose angles  $\alpha_i$  satisfy  $2\alpha_1 + 2\alpha_2 + 4\alpha_3 = 2\pi$ . However, we chose the angles so that  $\Sigma\alpha_i$  is not in  $\mathbb{Q}\pi$ ; hence this triangle admits no tiling with compact fundamental domain (cf. [20]). However this triangle does admit tilings with an infinite cyclic symmetry. The triangle is *weakly aperiodic*.

## 1 Introduction

As in [13], we consider the decidability of the Domino Problem in the hyperbolic plane— *Is there an algorithm to determine whether any given set of tiles admits a tiling?* In the Euclidean plane, R. Berger [2, 26] showed the Domino Problem is in fact undecidable, incidentally giving the first aperiodic sets of tiles in  $\mathbb{E}^2$ . R.M. Robinson considered the problem within  $\mathbb{H}^2$  [27] with limited success, and this question is not yet settled today.

The machinery of “regular production systems” is designed to capture the combinatorial structure of tilings. In [13], the subtlety of such systems, and of the Domino Problem itself, was highlighted by the construction of a “strongly aperiodic” set of tiles in the hyperbolic plane. Yet these systems are a quite practical tool for the construction of a wide variety of specific tilings, as we hope to demonstrate here through several examples and applications. In particular, we will discuss which triangles do, and which don’t, admit a tiling of  $\mathbb{H}^2$ ,  $\mathbb{E}^2$ , and  $\mathbb{S}^2$ . We give necessary and sufficient conditions for tiling on a measure one set of triangles and show a measure one set of triangles that do tile are weakly aperiodic.

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We begin by informally outlining the general method. In essence, the technique here is not that far from Poincaré’s own construction [18] or the more recent [20]. By explicitly considering the production systems abstractly, however, we gain additional information about the corresponding spaces of tilings, such as the existence of tilings that have an infinite cyclic symmetry.

Without belaboring the point, we assume polygons are closed topological disks endowed with locally finite sets of vertices and edges; a geometric polygon is embedded in  $X = \mathbb{H}^2, \mathbb{E}^2$  or  $\mathbb{S}^2$  and has piecewise smooth boundary. A **vertex arrangement**, a template for the polygons fitting around a vertex in a tiling, is properly defined as a cyclic collection of oriented edge pairings. One may make this more solid by adapting the machinery surrounding modern proofs of the Poincaré Fundamental Polygon Theorem (cf. [1], [10], [22]).

We define a  **$\mathcal{V}$ -complex**: Let  $P$  be any set of abstract polygons, with labeled edges and vertices, and let  $\mathcal{V}$  be a set of vertex arrangements of  $P$ . Then a connected 2-complex with polygonal faces is a  $\mathcal{V}$ -complex if and only if:

- (1) the complex has no boundary and is simply connected;
  - (2) each polygon is combinatorially a copy of one of the polygons in  $P$ ;
  - (3) each vertex of the complex is combinatorially a copy of one of the vertex arrangements in  $\mathcal{V}$ .
- In short, a  $\mathcal{V}$ -complex is a kind of abstract tiling by  $P$ , using the vertex arrangements  $\mathcal{V}$ .

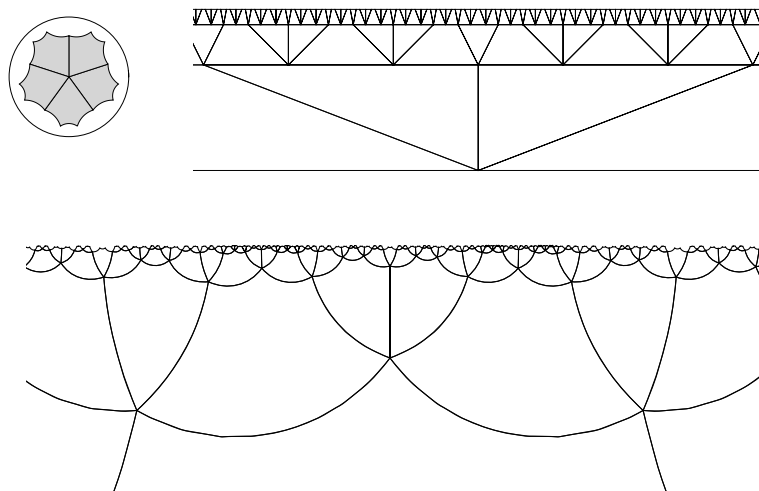


Figure 2: As an example, on the upper left, a vertex arrangement  $v$  of five (geometric) regular pentagons meeting meeting at a vertex. On the upper right, a portion of  $\mathcal{V}$ -complex of (abstract) pentagons,  $\mathcal{V} = \{v\}$ ; on the bottom, a portion of the tiling formed by charting the geometry.

**Lemma 1.1** *Suppose  $P$  is a set of geometric polygons in  $X = \mathbb{H}^2, \mathbb{E}^2, \mathbb{S}^2$ , and let  $\mathcal{V}$  be a set of vertex arrangements of  $P$ . If there exists a  $\mathcal{V}$ -complex, then  $P$  admits a tiling of  $X$ .*

More to the point, perhaps, this tiling is simply the  $\mathcal{V}$ -complex itself, inheriting the geometry of the tiles in  $P$ :

**Proof** This follows immediately from the observation (cf. [16], [25]) that the only simply-connected, complete Riemannian surfaces of constant curvature are  $\mathbb{H}^2, \mathbb{E}^2, \mathbb{S}^2$ . Simply note that the very definition of a  $\mathcal{V}$ -complex produces all these conditions; the geometry is charted by the geometry of the copies of  $P$  and the vertex arrangements.  $\square$

Our main technique then will be to show the existence of these  $\mathcal{V}$ -complexes.

Let  $P$  be a set of geometric tiles in  $\mathbb{H}^2$  or  $\mathbb{E}^2$ , admitting vertex arrangements  $\mathcal{V}$ . We will derive an alphabet, describing configurations by tiles in  $P$ , and a regular language describing locally embedded strips of these configurations lying along a curve.

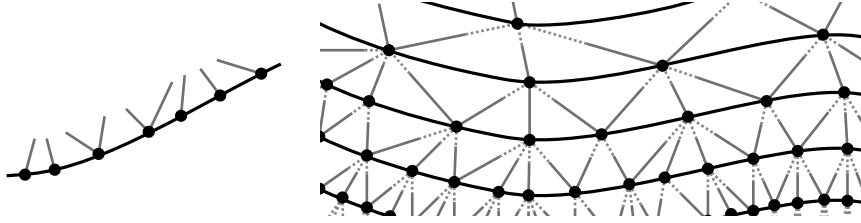


Figure 3: A schematic of our techniques: to a given set of tiles we associate an alphabet, corresponding to possible configurations along a curve, and a regular language corresponding to allowed sequences of these configurations, or strips of tiles, as at left. In the corresponding language of bi-infinite words we define a relation: two words are related if and only if the corresponding strips fit together, as at right. An orbit under this relation thus corresponds to an abstract complex with the combinatorial structure of a tiling by our original tiles.

We next derive a relation on words in our language. This relation is a generalization of a substitution on the letters; however, we do not end up with a function on the language— a given word may be related to one, no, or several other words.

This relation has a straightforward geometric meaning: one word is related to another if and only if the corresponding strips of tiles fit together locally.

Our relation can be extended to a corresponding set of bi-infinite strings—those for which every finite substring is a substring of some word in our regular language.

It will be far from clear, in general, whether there exists an orbit— a bi-infinite sequence of our bi-infinite words, each related to the next. Indeed, this is certainly undecidable under certain circumstances (cf. Section 2.2) and these relations seem to be interesting in their own right. But often, in particular applications, we are able to show the existence of orbits.

Now the bi-infinite words in the system correspond to abstract, infinite strips of tiles (abstract in that we no longer are concerned about these strips embedding) and two words are related iff the corresponding strips may fit together, abstractly. Consequently, if there exists an orbit, then we may construct a complex with precisely the combinatorial structure of a tiling by our original tiles, and by charting the geometry, we have in fact constructed a tiling.

The correspondence more or less goes the other way as well: given a  $\mathcal{V}$ -complex, we can (perhaps in a great many ways) produce an orbit in the corresponding regular production system. However, this converse requires more precise formulation— in general, modulo symmetry, orbits correspond to both a choice of tiling and a choice of a point on the sphere at infinity with respect to this tiling. However this converse correspondence is poorly understood.

Finally, we may turn the method around— beginning with a regular production system, ask, what geometry, if any, can be realized. The question takes some care to formulate, but stated carefully is almost certainly undecidable. On the other hand, this means that regular production systems can give rise to rich geometrical structures and provide a compelling model of growth and form (cf. [6], [23]).

The active reader may enjoy the *Mathematica* notebook “*Drawing Triangles in the Hyperbolic Plane*” [14], in which one may examine many of the specific constructions of this paper.

## 2 Regular productions

We take the convention that 0 is not a natural number, that  $\mathbb{N} = \{1, 2, \dots\}$ . We use [11] for standard definitions regarding languages. Let  $\mathcal{A}$  be any finite alphabet and  $\mathcal{L} \subset \mathcal{A}^*$  be any language on  $\mathcal{A}$ . Generally  $\mathcal{L}$  will be regular.

For any word  $\omega$ , let  $[\omega]$  be the length of  $\omega$ . We define the language  $\mathcal{L}^\infty \subset \mathcal{A}^\mathbb{Z}$  of **infinite words** to be sequences  $\omega \in \mathcal{A}^\mathbb{Z}$  such that every finite subsequence  $\omega(i) \dots \omega(j)$  is a subword of some word in  $\mathcal{L}$ . In general,  $\mathcal{L}^\infty$  may be empty. However, if  $\mathcal{L}$  is an infinite regular language,

by the Pumping Lemma (cf. any standard reference), then  $\mathcal{L}^\infty \neq \emptyset$ . Let  $\zeta : \mathcal{A}^\mathbb{Z} \rightarrow \mathcal{A}^\mathbb{Z}$  be the usual shift map,  $(\zeta(\omega))(i) = \omega(i-1)$ . We will often write  $\omega_i$  for  $\omega(i)$ , and on rare occasions write  $\omega_{i\dots j}$  for the word  $\omega(i)\dots\omega(j)$ .

Given an infinite set  $\{\sigma_n\}_{n \in \mathbb{Z}} \subset \mathcal{A}^*$  a word  $\omega \in \mathcal{A}^\mathbb{Z}$  is the **infinite concatenation**  $\dots\sigma_{-1}\sigma_0\sigma_1\dots$  iff for all  $n$   $\omega(a_{(n-1)+1})\dots\omega(a_n) = \sigma_n$  where  $a_0 = 0$ , for  $n > 0$ ,  $a_n = \sum_1^n \llbracket \sigma_i \rrbracket$  and for  $n < 0$ ,  $a_n = -\sum_{n+1}^0 \llbracket \sigma_i \rrbracket$ . This definition coincides with what one might expect, taking  $\omega(1)$  to coincide with  $\sigma_1(1)$ .

A **production relation**  $\mathcal{R} \subset (\mathcal{L} \times \mathcal{L}) \cup (\mathcal{L}^\infty \times \mathcal{L}^\infty)$  satisfies:<sup>2</sup>

1. There is a finite set  $\mathcal{R}_0 \subset (\mathcal{A} \times \mathcal{L})$  of “replacement rules”, and  $\mathcal{R}_0 \subset \mathcal{R}$ .
2. For any  $\omega, \sigma \in \mathcal{L}$ ,  $(\omega, \sigma) \in \mathcal{R}$  if and only if there exists  $\{\rho_i\}_{i \in \mathbb{Z}} \subset \mathcal{L}$  with  $(\omega(i), \rho_i) \in \mathcal{R}_0$  and  $\sigma = \rho_1 \dots \rho_{\llbracket \omega \rrbracket}$ .
3. For any  $\omega, \sigma \in \mathcal{L}^\infty$ ,  $(\omega, \sigma) \in \mathcal{R}$  if and only if there exist  $\{\rho_i\}_{i \in \mathbb{Z}} \subset \mathcal{L}$  and integer  $j$ ,  $0 \leq j < \llbracket \rho_0 \rrbracket$  such that for all  $j \in \mathbb{Z}$ ,  $(\omega(i), \rho_i) \in \mathcal{R}_0$  and  $\sigma = \zeta^j(\dots\rho_{-1}\rho_0\rho_1\dots)$  (in other words,  $\sigma(0)$  lies somewhere within  $\rho_0$ ).

For  $(\omega, \sigma) \in \mathcal{R}$ , we will write  $\omega \mapsto \sigma$  and say “ $\omega$  produces  $\sigma$ ”. Though the notation suggests that the relation is a function, it is not: a given word may be related to one, several, or no other words. A **production system**  $(\mathcal{A}, \mathcal{L}, \mathcal{R})$  is specified by an alphabet  $\mathcal{A}$ , language  $\mathcal{L}$  on  $\mathcal{A}$  and production relation  $\mathcal{R}$  on  $\mathcal{L} \cup \mathcal{L}^\infty$ .

Let  $\mathcal{P}$  be any property of a language  $\mathcal{L}$ . Then any production system  $(\mathcal{A}, \mathcal{L}, \mathcal{R})$  is a  **$\mathcal{P}$ -production system**. In particular, we will be considering *free* and *regular* production systems.

An **orbit** in a production system  $(\mathcal{A}, \mathcal{L}, \mathcal{R})$  is any set  $\{(\omega^i, j_i)\}_{i \in \mathbb{Z}} \subset \mathcal{L}^\infty \times \mathbb{N}$  such that for all  $i \in \mathbb{Z}$ ,  $(\omega^i, \omega^{i+1}) \in \mathcal{R}$ , with shift  $\zeta^{j_i}$ . An orbit is **periodic** if and only if there is some  $n \geq 1$  with  $\omega^i = \omega^{i+n}$ ,  $j_i = j_{i+n}$  for all  $i$ , and the period of the orbit is the minimal such  $n$ .

For a production system  $(\mathcal{A}, \mathcal{L}, \mathcal{R})$ , inductively define the set of **superwords**  $\Sigma(\mathcal{A}, \mathcal{L}, \mathcal{R}) \subset \mathcal{L}$ :

- (a) For each  $(\omega, \sigma) \in \mathcal{R}_0$ ,  $\omega \in \Sigma(\mathcal{A}, \mathcal{L}, \mathcal{R})$  and
- (b) if  $\omega \in \Sigma(\mathcal{A}, \mathcal{L}, \mathcal{R})$  and  $(\omega, \sigma) \in \mathcal{R}$ , then  $\sigma \in \Sigma(\mathcal{A}, \mathcal{L}, \mathcal{R})$ .

The set of **infinite superwords** is defined as  $\Sigma^\infty(\mathcal{A}, \mathcal{L}, \mathcal{R}) := (\Sigma(\mathcal{A}, \mathcal{L}, \mathcal{R}))^\infty \subset \mathcal{L}^\infty$ .

**Example 2.1** We pause for a simple example: Let  $\mathcal{A} = \{0, 1, 2\}$  and let  $\mathcal{L}$  the regular language consisting of all subwords of  $L((012(12)^*)^*)$ . Thus the word 01201212 is in our  $\mathcal{L}$ , but 121001 is not. For  $\mathcal{R}_0$ , we take  $0 \mapsto 12$ ,  $1 \mapsto 12$ ,  $1 \mapsto 21$ , and  $2 \mapsto 01$ ,  $2 \mapsto 20$ . Note that the relation  $\mathcal{R}$  on  $\mathcal{L}$  is not a function: For no  $\sigma \in \mathcal{L}$  does  $0120 \mapsto \sigma$  hold. If  $012120 \mapsto \sigma$ , we must have  $\sigma = 121201212012$ . But  $12 \mapsto 1212$  and  $12 \mapsto 2120$ . Here  $\Sigma^\infty$  consists only of the shifts of  $\dots 0121201212012120\dots$  and there indeed are periodic orbits, as we will show in Example 2.10.

## 2.1 Symbolic substitution systems

A **symbolic substitution system** is a free production system such that for each  $a \in \mathcal{A}$ , there is exactly one  $\sigma \in \mathcal{R}_0$  with  $(a, \sigma) \in \mathcal{R}_0$ , and such that  $\Sigma^\infty \neq \emptyset$ . A symbolic substitution system is **primitive** if and only if, for some  $n$ , for every  $a, b \in \mathcal{A}$ , the letter  $b$  occurs within  $\mathcal{R}^n(a)$ . Since here  $\mathcal{R}$  is in fact a function  $\mathcal{A}^* \rightarrow \mathcal{A}^*$ , we will write  $\mathcal{R}(\omega) = \sigma$  for  $(\omega, \sigma) \in \mathcal{R}$ .

These definitions of symbolic substitution systems coincide with the usual ones, on bi-infinite strings (cf. [24]), except for the use of shifts in the definition of the map. This difference plays a key role in the cardinality of the set of orbits in the Proposition below, as the example  $\mathcal{A} = \{0\}$ ,  $\mathcal{L} = \mathcal{A}^*$ ,  $\mathcal{R} = \{(0, 00)\}$  illustrates— even though there is only one word in  $\Sigma^\infty$ , there are uncountably many ways in which it can be decomposed into larger and larger superwords, and uncountably many distinct orbits (see Section 3.1). In effect, we are really describing one-dimensional substitution tiling systems.

<sup>2</sup>Somewhat more general axioms are available by taking  $\mathcal{R}_0 \subset (\mathcal{L} \times \mathcal{L})$  with appropriate changes elsewhere.

The following might be considered a folk theorem; variations appear in [24] (for substitutions on one-sided sequences) and [8] (for substitution tilings).

**Proposition 2.2** *In any symbolic substitution system there are uncountably many orbits, countably many of which are periodic.*

**Proof** Let  $(\mathcal{A}, \mathcal{A}^*, \mathcal{R})$  be a symbolic substitution system. For each  $\omega \in \mathcal{A}^*$ ,  $i \in \{2, \dots, \llbracket \omega \rrbracket\}$ , define  $\eta(\omega, i) = \sum_{k=1}^{i-1} \llbracket \mathcal{R}(\omega_k) \rrbracket$ ; that is,  $\eta(\omega, i)$  is the position just before  $(\mathcal{R}(\omega_i))_1$  in  $\mathcal{R}(\omega)$ . Take  $\eta(\omega, 1) = 0$ .

By a pigeonhole argument, there must exist a letter  $a \in \mathcal{A}$  and an  $n \in \mathbb{N}$  such that  $a$  appears twice in the interior of the word  $\mathcal{R}^n(a)$ , say at indices  $u, v$ ,  $1 < u, v < \llbracket \mathcal{R}^n(a) \rrbracket$ ,  $u \neq v$ . Without any loss of generality we may assume  $n = 1$ . For each  $\alpha \in \{u, v\}^{\mathbb{Z}}$  we will construct an orbit; this orbit will be periodic if and only if  $\alpha$  itself has a period under  $\zeta$ .

Let  $A^i$  denote  $\mathcal{R}^i(a)$ . For each  $j \in \mathbb{Z}$ , we define  $s(0, j) = 1$  and for  $i \in \mathbb{N}$ , inductively let

$$s(i, j) = \eta(A^i, s(i-1, j)) + \alpha_{i+j}$$

Note for all  $j \in \mathbb{Z}$ ,  $i \geq 0$ , that  $A^i = (A^{i+1})_{(k+1)\dots(k+\llbracket A^i \rrbracket)}$  where  $k = s(i+1, j-1) - s(i, j)$ . For each  $j \in \mathbb{Z}$ , we consider the word  $\omega^j$  defined by specifying that, for each  $i \geq 0$ ,

$$\zeta^{s(i, j-i)}(\omega^j)_{1\dots\llbracket A^i \rrbracket} = A^i$$

It is a matter of notation to check that each  $\omega^j$  is well defined and that  $\mathcal{R}(\omega^j) = \omega^{j+1}$  with shift  $\llbracket A^1 \rrbracket + 1 - \alpha_j$ . We have constructed an orbit corresponding to  $\alpha$  as promised. We thus have uncountably many orbits, infinitely many of which are periodic. In fact, there can only be countably many periodic orbits: a periodic orbit  $\{\omega^i, j_i\}$  is completely specified by a finite list of shifts and the letter  $\omega_0^0$ .  $\square$

As described in the next section, for an arbitrary regular production system, we cannot be sure whether such orbits exist, or if so, whether there must be a periodic orbit, in sharp contrast to the simplicity of substitution production systems. The techniques of Section 2.3 will allow us to construct orbits for the systems we will use in this paper.

## 2.2 Decidability, aperiodicity and growth

A production relation has **asymptotic growth rate**  $\alpha$  if and only if for any  $\epsilon > 0$ , there is a length  $n$  such that for all  $\omega, \sigma$ , with  $\llbracket \omega \rrbracket > n$  and  $\omega \mapsto \sigma$ , we have  $|\alpha \llbracket \omega \rrbracket - \llbracket \sigma \rrbracket| < \epsilon \llbracket \omega \rrbracket$ . A production relation is **linear** if and only if it has asymptotic growth rate 1, and **strongly expansive** if and only if it has asymptotic growth rate greater than 1.

Berger's celebrated result that, in the Euclidean plane, the "Domino Problem" is undecidable and that there exist aperiodic sets of tiles [2], can be interpreted as:

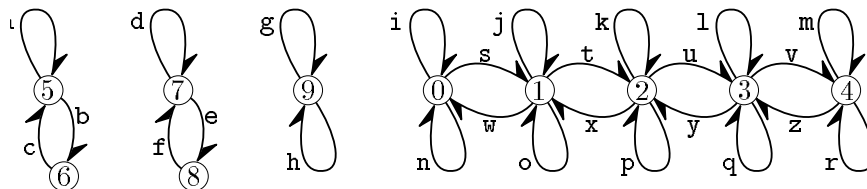
**Theorem 2.3 (Berger)** *There is no algorithm to determine whether a linear regular production system has an orbit. There exist linear regular production systems for which there is an orbit but no periodic orbit.*

What if the system is not linear, and words in the production expand rapidly? Is there an algorithm to determine whether or not there exist orbits? It is unclear which paradigm reigns, the explosive freedom of free production systems or the tight constraints of the Euclidean tilings.

We do not know if it is decidable whether or not a strongly expansive regular production system has an orbit. But the example of a strongly aperiodic protoset in  $\mathbb{H}^2$ , given in [13], can be interpreted as:

**Theorem 2.4** *There exists a strongly expansive regular production system for which there are orbits, but no periodic orbits.*

This contrasts with the simplicity of symbolic substitution systems, as seen in Proposition 2.2. In [13], in effect, we show that the following rather mysterious system has orbits, but no periodic orbits:  $\mathcal{A} = \{a, b, c, \dots, x, y, z\}$ . The language  $\mathcal{L}$  is given by the graphs



The 78 productions are

$$\begin{aligned}
 a, d &\rightarrow ggh & b, c &\rightarrow hhh & e, f &\rightarrow ggg \\
 g &\rightarrow i, j, k, l, m, s, t, u, v & i, j, k, l, m, w, x, y, z &\rightarrow a, b, e \\
 h &\rightarrow m, o, p, q, r, w, x, y, z & n, o, p, q, r, s, t, u, v &\rightarrow c, d, f
 \end{aligned}$$

In effect, regular production systems with asymptotic growth rates are precisely those that can model tilings of constant curvature. We might turn the question around and ask then, which production systems do in fact have such rates? The following is well-known [24]:

**Proposition 2.5** *Any primitive symbolic substitution system has an (easily calculated) asymptotic growth rate.*

On the other hand, this is almost certainly the case:

**Conjecture 2.6** *There is no algorithm to determine whether any given regular production has an asymptotic growth rate.*

The author strongly suspects that a proof of this theorem can be made by modifying an incarnation of the productions of Post (cf. [21]): given some Post production  $X$ , create a new production system  $Y$  that has an asymptotic growth rate if and only if  $X$  has a periodic orbit (i.e. enters a loop). As this is undecidable, so to is question of whether  $Y$  is expansive.

**Example 2.7 (The Kolakoski sequence)** A variation on the Kolakoski sequence provides a nice example of the subtlety of growth rates. Let  $\mathcal{A} = \{1, 2, \cdot\}$ ; let  $\mathcal{L}$  consist of words of the form  $((1 + 11) \cdot (2 + 22) \cdot)^*$ ; and let  $\mathcal{R}$  be given by

$$1 \mapsto 1 \cdot \quad 1 \mapsto 2 \cdot \quad 2 \mapsto 11 \cdot \quad 2 \mapsto 22 \cdot \quad \cdot \mapsto \epsilon$$

where  $\epsilon$  is the empty word. The existence of orbits is not difficult to establish: there are uncountably many, countably many of which are periodic. The word

$$\omega = \dots 11 \cdot 22 \cdot 1 \cdot 22 \cdot 1 \cdot 2 \cdot 11 \cdot 22 \cdot 1 \cdot 22 \cdot 11 \cdot 2 \cdot 1 \cdot 22 \cdot 1 \cdot 22 \cdot 11 \dots$$

has  $\omega \mapsto \omega$ , where  $\omega_0$  is taken to be the marked ‘.’. The Kolakowski sequence is the right half of this word. It is a well-known open question whether this production system has asymptotic growth rate  $3/2$ , that is, whether there are asymptotically as many 1’s as 2’s [17].

### 2.3 Finding orbits

In general, given an arbitrary regular production system  $(\mathcal{A}, \mathcal{L}, \mathcal{R})$ , we have no understanding of the existence of orbits in  $\mathcal{L}^\infty$  under  $\mathcal{R}$ —and indeed there may well be no algorithm to determine whether such orbits exist.

Here we provide two simple tests that—when successful—can be used to show, that some given regular production system has periodic orbits. However, one must note that in general,  $a$

*priori*, a touch of magic is necessary to apply these tests, and indeed, they cannot be used when no periodic orbits exist—such as in the example of [13].

Let  $(\mathcal{A}, \mathcal{L}, \mathcal{R})$  be a regular production system. A symbolic substitution system  $(\mathcal{A}^-, (\mathcal{A}^-)^*, \mathcal{R}^-)$  is a **cut** of  $(\mathcal{A}, \mathcal{L}, \mathcal{R})$  iff

- (a)  $\mathcal{A}^- \subset \mathcal{A}$ ,
- (b)  $\mathcal{R}^- \subset \mathcal{R}$  and
- (c)  $\Sigma((\mathcal{A}^-)^*, \mathcal{R}^-) \subset \mathcal{L}$ .

Consider any two regular production systems  $(\mathcal{A}_0, \mathcal{L}_0, \mathcal{R}_0)$ , and  $(\mathcal{A}_1, \mathcal{L}_1, \mathcal{R}_1)$ . Any map  $\phi : \mathcal{A}_0 \rightarrow \mathcal{A}_1$  induces maps from  $\phi^* : \mathcal{A}_0^* \rightarrow \mathcal{A}_1^*$  and  $\phi^{**} : \mathcal{A}_0^* \times \mathcal{A}_0^* \rightarrow \mathcal{A}_1^* \times \mathcal{A}_1^*$  in the obvious manner. Thus:

A symbolic substitution system  $(\mathcal{A}^+, (\mathcal{A}^+)^*, \mathcal{R}^+)$ , with a map  $\phi : \mathcal{A}^+ \rightarrow \mathcal{A}$  is a **lift** of  $(\mathcal{A}, \mathcal{L}, \mathcal{R})$  iff

- (a)  $\phi^{**}(\mathcal{R}^+) \subset \mathcal{R}$  and
- (b) there is a regular language  $\mathcal{L}^+ \subset (\mathcal{A}^+)^*$  such that  $\Sigma((\mathcal{A}^+)^*, \mathcal{R}^+) \subset \mathcal{L}^+$  and  $\phi^*(\mathcal{L}^+) \subset \mathcal{L}$ .

**Lemma 2.8** *Let  $(\mathcal{A}, \mathcal{L}, \mathcal{R})$  be a regular production system with a lift or a cut. Then there are orbits, and in particular, there are periodic orbits, in  $\mathcal{L}^\infty$  under  $\mathcal{R}$ .*

The proof is essentially a trivial exercise in notation:

**Proof** Suppose there exists a cut  $(\mathcal{A}^-, (\mathcal{A}^-)^*, \mathcal{R}^-)$  of  $(\mathcal{A}, \mathcal{L}, \mathcal{R})$ . Now there exists a periodic orbit in  $\Sigma^\infty(\mathcal{A}^-, (\mathcal{A}^-)^*, \mathcal{R}^-) \subset ((\mathcal{A}^-)^*)^\infty$ . But also,  $\Sigma(\mathcal{A}^-, (\mathcal{A}^-)^*, \mathcal{R}^-) \subset \mathcal{L}$  and so  $\Sigma^\infty(\mathcal{A}^-, (\mathcal{A}^-)^*, \mathcal{R}^-) \subset \mathcal{L}^\infty$ .

Now  $\mathcal{R}^- \subset \mathcal{R}$  and consequently  $(\mathcal{R}^-)^\infty \subset \mathcal{R}$ . And so we are done: any orbit in  $\Sigma^\infty$  is actually an orbit in  $\mathcal{L}^\infty$  under  $\mathcal{R}$ .

The other case, when there is a lift, goes in similar fashion. □

The following trivial lemma is a useful tool when demonstrating that we have a cut or a lift.

**Lemma 2.9** *Let  $(\mathcal{A}, \mathcal{A}^*, \mathcal{R})$  be a symbolic substitution system. Let  $\mathcal{L} \subset \mathcal{A}^*$ . If  $\mathcal{R}(\mathcal{L}) \subset \mathcal{L}$ , and  $\mathcal{A} \subset \mathcal{L}$ , then  $\Sigma \subset \mathcal{L}$ .*

**Example 2.10** Let  $\mathcal{A}, \mathcal{L}, \mathcal{R}$  be as in Example 2.1. Let  $\mathcal{A}^+ = \{0, 1, \mathbf{1}, 2, \mathbf{2}\}$ , with the obvious map onto  $\mathcal{A}$ :  $\phi(0) = 0, \phi(1) = \phi(\mathbf{1}) = 1, \phi(2) = \phi(\mathbf{2}) = 2$ . Let  $\mathcal{L}^+$  consist of all subwords of  $L(0\mathbf{1}\mathbf{2}\mathbf{1}2)$ . Take as our rules in  $\mathcal{R}^+$ :

$$0 \mapsto \mathbf{1}\mathbf{2} \quad 1 \mapsto \mathbf{2}\mathbf{1} \quad \mathbf{1} \mapsto 1\mathbf{2} \quad 2 \mapsto 2\mathbf{0} \quad \mathbf{2} \mapsto 0\mathbf{1}$$

Note of course that  $\phi^{**}(\mathcal{R}^+) \subset \mathcal{R}$ . Now  $\mathcal{R}(\mathcal{L}^+)$  consists of all subwords of  $L(\mathbf{1}\mathbf{2} \mathbf{1}\mathbf{2} 0\mathbf{1} \mathbf{2}\mathbf{1} 2\mathbf{0})^*$ , and lies in  $\mathcal{L}^+$ . Hence,  $\Sigma((\mathcal{A}^+)^*, \mathcal{R}^+) \subset \mathcal{L}^+$ . Applying Lemma 2.9, our original system does in fact have periodic orbits on infinite superwords.

But again, one shouldn't expect these techniques to work in general. A corollary of Theorem 2.4 is that there exist regular production systems for which there are no cuts or lifts.

### 3 Geometric examples

We are now ready to illustrate the method with new constructions in  $\mathbb{H}^2$ . In each case, we construct a regular production system that captures the local combinatorics of a desired tiling. We then find a lift or a cut of the system to construct orbits, and consequently a complex with the desired combinatorics; finally, appealing to Lemma 1.1, we have the desired tiling. In the interest of brevity, we are only sketching the constructions in this section.

### 3.1 Symbolic substitution tilings

This example, due to L. Sadun, nicely illustrates the connection between these productions and tilings. To every primitive symbolic substitution  $(\mathcal{A}, \mathcal{A}^*, \mathcal{R})$  system there exists a set  $T$  of tiles in  $\mathbb{H}^2$  such that: (1) in any tiling by the tiles in  $T$ , the tiles lie in horocyclic rows, (2) there is a bijection  $\phi : T \rightarrow \mathcal{A}$ ; (3) extending this map  $\phi$  to take bi-infinite strips of tiles in horocyclic rows to words in  $\mathcal{A}^\infty \bmod \zeta$ , any bi-infinite strip in any tiling by  $T$  will be mapped to an infinite superword; and (4) in any tiling, one horocyclic row  $r_1$  lies directly “above” another row  $r_2$  if and only if  $\phi(r_1) \mapsto \phi(r_2)$ . That is, each orbit gives rise to a tiling by the tiles; periodic orbits give rise to tilings with an infinite cyclic symmetry.

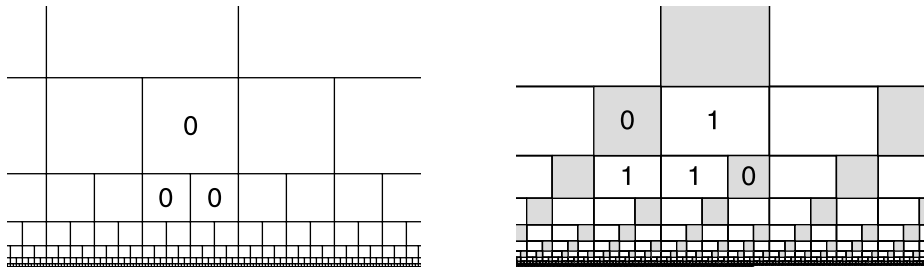


Figure 4: A set of tiles, and tilings corresponding to the primitive substitution systems  $0 \mapsto 00$  (left) and to the primitive substitution  $0 \mapsto 1, 1 \mapsto 10$  (right). In both cases there are uncountably many tilings by these tiles, precisely corresponding to the orbits in the substitution system.

We have some flexibility in the construction: we may change the horizontal scale of our illustrations— thus applying a non-quasi-isometric map that preserves the combinatorial structure of the tilings (see Section 3.3 and Figure 8).

### 3.2 Aperiodic Archimedean tilings

Following B. Grünbaum, we call a tiling by regular polygons **Archimedean** if each vertex link is congruent. This is weaker than being **uniform**, in which there is a symmetry of the tiling as a whole that acts transitively on the vertices. Consequently, there can be non-periodic, but Archimedean, tilings!

The following example is due to Č. Marek [19], though he did not have a formal apparatus to prove the tiling really exists globally. In a forthcoming paper, N. Wetzler and this author extend this example to give a vast collection of polygons that admit only non-periodic Archimedean tilings.

**Theorem 3.1** *There exists (uncountably many) Archimedean tilings of  $\mathbb{H}^2$  in which three regular pentagons and one equilateral triangle meet at each vertex. No such tiling admits a co-compact symmetry, though countably many admit an infinite cyclic symmetry.*

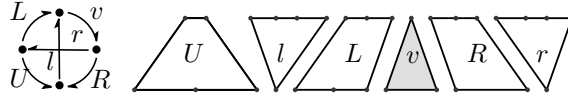
In essence, the proof simply consists of constructing an appropriate regular production system.

**Proof** An elementary calculation reveals that there are unique angles  $\alpha, \beta$  and length  $l$  such that  $3\alpha + \beta = 2\pi$  and the regular pentagon with vertex angles  $\alpha$ , and the regular triangle with vertex angles  $\beta$  both have edge length  $l$ . Moreover, as linear combinations of these angles are incommensurate with  $\pi$ , no finite collection of such pentagons and triangles can be a fundamental domain for a symmetrical tiling of  $\mathbb{H}^2$ . However, there is a tiling by such triangles and pentagons:

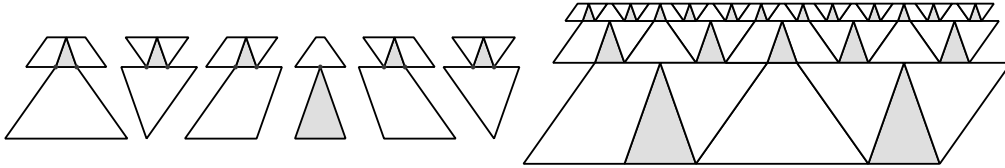
Consider  $\mathcal{A} = \{L, R, l, r, U, v\}$ , with language  $\mathcal{L}$  with words described by paths in the following graph (one may begin and end at any vertex). Each letter is in correspondence with an abstract, local combinatorial arrangement of tiles, as illustrated. A word, then, describes a



sequence of such arrangements, strung along a curve. (The tiles in the illustration are designed to fit together if and only if the corresponding word is in our language.)



We take as rules  $U \mapsto LvR$   $l, r \mapsto rvl$   $L \mapsto Lvl$   $v \mapsto U$   $R \mapsto rvR$  with combinatorial interpretation:



On the right we see a complex corresponding to the production

$$LvlUrvl \mapsto LvlUrvlLvRrvlUrvl \mapsto LvlUrvlLvRrvlUrvlLvlUrvRrvlUrvlLvRrvlUrvl$$

An easy inductive argument shows that any  $\mathcal{V}$ -complex corresponding to any orbit in this production system has the combinatorial structure of an Archimedean tiling with three pentagons and one triangle at each vertex.

Similarly, it is easy to apply Lemma 2.9 and note that the superwords of the symbolic substitution system  $(\mathcal{A}, \mathcal{A}^*, \mathcal{R})$  lie in  $\mathcal{L}$ ; that is,  $(\mathcal{A}, \mathcal{L}, \mathcal{R})$  has a cut and so has orbits; consequently, there is a  $\mathcal{V}$ -complex with the desired structure. As there really do exist regular pentagons and triangles that can fit together locally in the correct manner, we can chart a geometry onto this complex and obtain a tiling.

As a final note, the uncountably many non-periodic orbits in the production system correspond to tilings with no infinite cyclic symmetry; the countably many periodic orbits correspond to tilings with an infinite cyclic symmetry. The regular pentagon and triangle with edge length  $l$  are thus *weakly aperiodic*.  $\square$

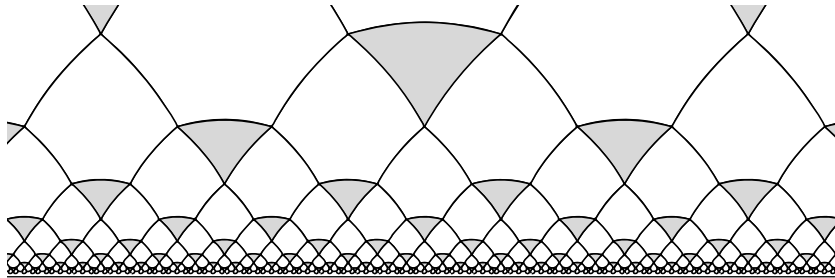


Figure 5: A non-periodic Archimedean tiling

### 3.3 Non-quasi-isometric maps on regular tilings

Let  $\{p, q\}$  denote the tiling by regular  $p$ -gons with vertex angles  $2\pi/q$ . A few years ago, L. Danzer made a strange observation [7], which clearly generalizes:

If we delete an edge in the  $\{5, 4\}$  tiling, we replace two pentagons with a (very distorted!) octagon. If we can manage to delete infinitely many edges, exactly one meeting each vertex of the tiling, we will replaced our  $\{5, 4\}$  tiling with a (very distorted!) tiling by octagons meeting three-to-a-vertex.

Conversely, we can split an octagon by adding an edge that cuts it into two (very distorted!) pentagons. If we can manage to add infinitely many edges to the  $\{8, 3\}$  tiling, exactly one

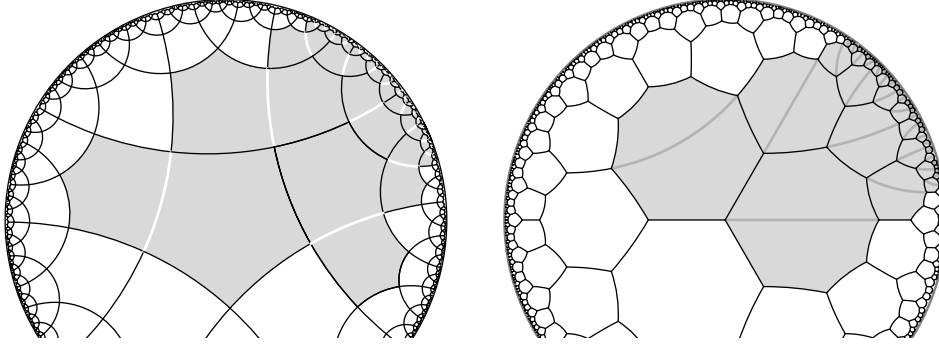


Figure 6: The two gray regions are homeomorphic: edges have been deleted from a portion of the  $\{5, 4\}$  tiling and added to a portion of the  $\{8, 3\}$  tiling. The region is tiled by pentagons meeting in fours, if we include the lightly colored edges, or by octagons meeting in threes, if we do not. Is there a global recipe for defining such a homeomorphism?

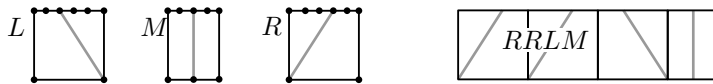
meeting each vertex of the tiling, we will have replaced our  $\{8, 3\}$  tiling with a (very distorted!) tiling by pentagons meeting four-to-a-vertex. In either case, we are trying to establish whether the graph of edges of the  $\{8, 3\}$  tiling is a sub-graph of the graph of edges of the  $\{5, 4\}$  tiling.

It is not difficult to see that *locally* this is the case; but Danzer asked whether there is a *global* recipe for deleting edges from the  $\{5, 4\}$  tiling, or adding edges to the  $\{8, 3\}$  tiling, or embedding the latter's graph of edges in that of the former. Danzer noted that if such global correspondence exists:

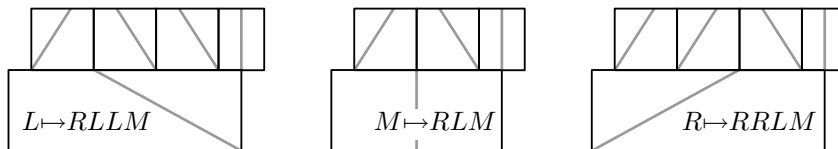
First, any such correspondence between the  $\{5, 4\}$  tiling and the  $\{8, 3\}$  would give rise to a map from the hyperbolic plane to itself; however this map could not be a quasi-isometry: right-angled pentagons have area  $\pi/2$  and the octagons in the  $\{8, 3\}$  tiling have area  $2\pi/3$ , and distances would be arbitrarily distorted. No such map could be well-behaved with respect to the symmetries of the tilings: no co-compact subgroup of the symmetries of the  $\{5, 4\}$  tiling is a subgroup of the symmetries of the distorted  $\{8, 3\}$  tiling.

But the main puzzle is simply: *Can the local condition be satisfied globally?* Can exactly one edge be deleted at each vertex of the  $\{5, 4\}$  tiling?

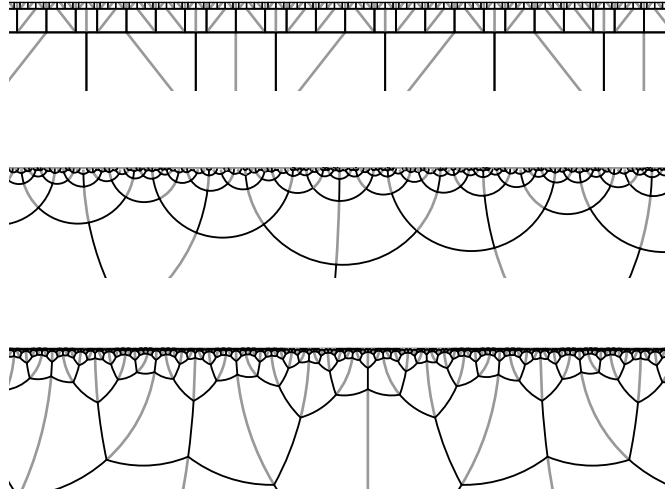
To answer Danzer's question, we simply construct a regular production system capturing the desired combinatorics. We take as our alphabet  $\mathcal{A} = \{L, M, R\}$ , interpreted as in the figure below; our language  $\mathcal{L}$  consists of all words in which  $L$  is not immediately followed by  $R$ .



The rules are  $L \mapsto LLMR$ ,  $M \mapsto LMR$  and  $R \mapsto LMRR$ , interpreted as:



Again it is an easy matter to show that there exist orbits, and that any complex corresponding to such an orbit has the correct combinatorics: ignoring the lightly shaded lines, we have a complex with the combinatorial structure of the  $\{8, 3\}$  tiling; including them, we have the structure of the  $\{5, 4\}$  tiling. We may chart either geometry (or any other we might please), establishing our homeomorphism.



It is striking that, even accounting for symmetries, there are uncountably many distinct ways to embed the graph of edges of the  $\{8, 3\}$  tiling into those of the  $\{5, 4\}$  tiling (each arising from the different orbits in the system). Moreover, countably many of these preserve an infinite cyclic subgroup of the symmetries of the tiling.

## 4 Triangles

We now turn to the application of regular production systems to tilings by copies of a single given triangle. We will view triangles in  $\mathbb{H}^2, \mathbb{E}^2, \mathbb{S}^2$  as ordered triples  $\alpha, \beta, \gamma \in \mathbb{R}^3$  of angles, criss-crossed by planes of the form  $r\alpha + s\beta + t\gamma = 2\pi$ ,  $r, s, t \in \mathbb{N}$ , dividing the space of triangles into a 3-complex. (The famous Poincaré triangles, with angles  $\pi/p, \pi/q, \pi/r$ , lie on discrete set of vertices in this complex.)

In effect, in Theorem 4.5, we will note that “admitting a tiling” is a property of the *cells* in this complex: that is, within a given cell either all the triangles do admit a tiling, or they do not.

Lemma 4.2 (trivially) points out that no triangle in the interior of a 3-cell can possibly admit a tiling. On the other hand, Theorem 6.2 gives necessary and sufficient conditions for a triangle in the interior of a 2-cell to admit a tiling.

We relegate many other cases to a set of notes “*Further triangle tilings*”, [15], where the behavior of triangles on various families of 1-cells is analyzed. The most interesting outcome may be that there is no algorithm to decide whether a given triangle admits a tiling, that undecidability lurks in the remaining nooks and crannies of this complex [12]. For what it’s worth, many cases have eluded this author, despite elaborate effort.

### 4.1 A space of triangles

Suppose  $T_1, T_2 \subset X_1, X_2 \in \{\mathbb{H}^2, \mathbb{E}^2, \mathbb{S}^2\}$  are triangles with the same vertex-angles, perhaps after reindexing the vertices. Then clearly  $X_1 = X_2$ , the triangles are similar, and  $T_1$  admits a tiling if and only if  $T_2$  does. Consequently, we will regard triangles not only in the usual sense— a convex three-sided polygon with geodesic edges— but also as points in the half open cube  $\mathcal{O} = \{(\alpha, \beta, \gamma) \mid 0 < \alpha, \beta, \gamma \leq \pi\} \subset \mathbb{R}^3$ . Note the correspondence goes the other way as well: for all  $(\alpha, \beta, \gamma) \in \mathcal{O}$ , applying a little trigonometry, we find a triangle in  $\mathbb{H}^2, \mathbb{E}^2$  or  $\mathbb{S}^2$  with vertex angles  $\alpha, \beta, \gamma$ , with the sole exception that there exist no scalene triangles with a vertex angle  $\pi$ .

Of course, it would be most natural to consider unordered triples of angles, and a triangle as a point in the quotient of  $\mathcal{O}$  under permutations of coordinates, but affine subspaces of  $\mathcal{O}$  play

a helpful and central role and for simplicity we do not take this quotient.

Consider the set  $\Pi$  of planes  $\pi_{rst} = \{(\alpha, \beta, \gamma) \mid r\alpha + s\beta + t\gamma = 2\pi\}$ ,  $r, s, t \in \mathbb{N}$ . Equilateral and isosceles triangles have a somewhat special combinatorial structure, and we also have to consider the planes on which these triangles lie: let  $\pi_{12}, \pi_{23}, \pi_{31}$  be the planes given by  $\alpha = \beta$ ,  $\beta = \gamma$ ,  $\alpha = \gamma$  respectively and  $\Pi^+ = \Pi \cup \{\pi_{12}, \pi_{23}, \pi_{31}\}$ . Of course, a triangle  $T$  is in  $\mathbb{S}^2, \mathbb{E}^2, \mathbb{H}^2$  iff the corresponding point of  $\mathcal{O}$  is above, on, or below the plane  $\pi_{222}$ . A triangle is scalene if and only if it does not lie on one of the planes  $\pi_{12}, \pi_{23}, \pi_{31}$ . As mentioned above, there are no scalene triangles on the planes  $\pi_{002}, \pi_{020}, \pi_{200}$ . One can easily verify, though Figure 7 may make one doubt:

**Lemma 4.1** *Any closed subset of  $\mathcal{O}$  intersects only finitely many planes in  $\Pi^+$ . In particular, we may regard  $\cup \Pi^+ \subset \mathcal{O}$  as a 3-complex.*

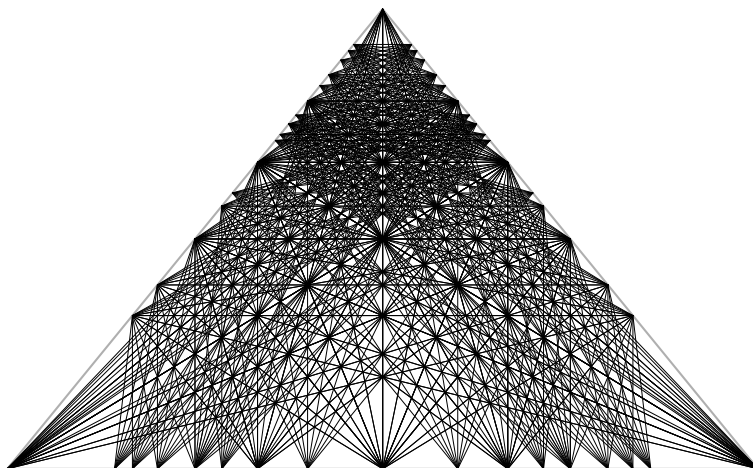


Figure 7: A slice of  $\cup \Pi$ : The plane  $\pi_{224} \cap \mathcal{O}$  and its intersections with a few planes in  $\Pi$ ; the full set of intersections is discrete in  $\mathcal{O}$ , but accumulates on  $\partial \mathcal{O}$ .

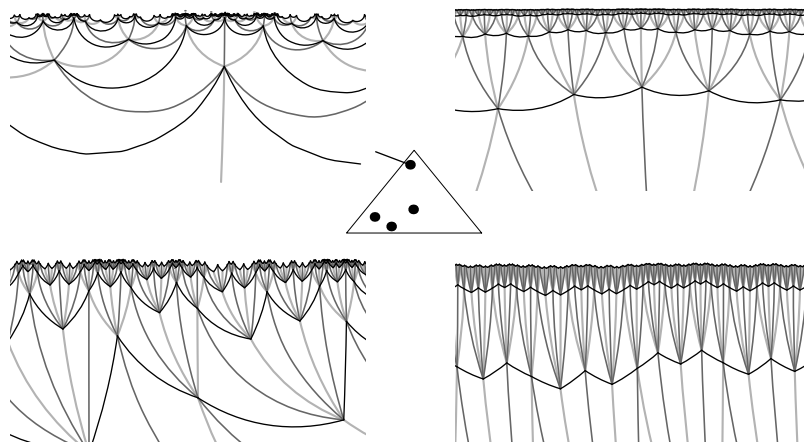


Figure 8: The four marked points in  $\pi_{224} \cap \mathcal{O}$  (center) correspond to four triangles in  $\mathbb{H}^2$ . Homeomorphic (“combinatorially equivalent”) tilings by these four triangles are shown. Note that in general area varies from one triangle to another and there is no quasi-isometry taking one of these homeomorphic tilings to the next. Each triangle  $T \in \overset{\circ}{\pi}_{224}$  admits uncountably many non-homeomorphic tilings; however for any tiling by  $T$ , for any triangle  $T' \in \pi_{224}$ , there is a homeomorphic tiling by  $T'$ .

This rather sad lemma settles, for a measure-1 set of triangle, the question of admitting a tiling or not; we do quite a bit better with a measure-1 subset of  $\cup\Pi$  in Theorem 6.2.

**Lemma 4.2** *Let  $\mathsf{T} \notin \cup\Pi$ ; then  $\mathsf{T}$  does not admit a tiling.*

**Proof** If  $\mathsf{T} \notin \cup\Pi$ , then for no  $r, s, t \in \mathbb{N}$  is  $r\alpha + s\beta + t\gamma = 2\pi$ . But then  $\mathsf{T}$  admits no vertex arrangements, much less tilings!  $\square$

Given a set of finite planes  $P$  of planes in  $\Pi$ , define the affine subspace  $\cap P = \{\mathsf{T} \in \mathcal{O} \mid \forall \pi \in P, \mathsf{T} \in \pi\}$ , and the set  $\overset{\circ}{\cap} P = \{\mathsf{T} \in \cap P \mid \forall \pi \notin P, \mathsf{T} \notin \pi\}$ . A set  $P$  of planes is **maximal** iff  $\overset{\circ}{\cap} P \neq \emptyset$ . (That is, maximal sets of planes precisely correspond to the affine subspaces in  $\cup\Pi$ .) From Lemma 4.1, it soon follows that:

**Lemma 4.3** *For every  $P \subset \Pi^+$ ,  $\cap P \neq \emptyset$ , there exists a unique maximal  $\tilde{P}$  with  $P \subset \tilde{P}$ ,  $\cap P = \cap \tilde{P}$ ; this  $\tilde{P}$  is finite.*

For example, if  $P = \{\pi_{402}, \pi_{060}\}$ ,  $\tilde{P} = \{\pi_{402}, \pi_{231}, \pi_{060}\}$ . If  $\cap P = \emptyset$ , define  $\tilde{P} = \emptyset$ . We will write

$$[P] := \text{“} \cap P \neq \emptyset \text{ and } \forall \mathsf{T} \in \cap \tilde{P}, \mathsf{T} \text{ admits a tiling”}$$

$$(P) := \text{“} \forall \mathsf{T} \in \overset{\circ}{\cap} \tilde{P}, \mathsf{T} \text{ does not admit a tiling”}$$

Note that for  $P \subset \Pi$ , the assertion  $(P)$  makes no claim of discussing isocoles or equilateral triangles. The following is trivial; the final statement follows from the observation that, under the hypotheses,  $\cap P = \cap \tilde{P} = \cap \tilde{Q} = \cap Q$ .

**Lemma 4.4** *If  $[P]$  then  $[P \cup Q]$ ;  $[P]$  if and only if  $[\rho P]$  for any permutation  $\rho$  of the coordinates of  $\mathcal{O}$ . If  $\tilde{P} = \tilde{Q}$  then  $[P]$  if and only if  $[Q]$ , if and only if  $[\tilde{P}]$ ,  $[\tilde{Q}]$ .*

The point of all this is this helpful theorem, which will follow as an immediate corollary of Lemma 4.11; our cases will thus be based on the structure of our sets  $P$ .

**Theorem 4.5** *For all  $P \subset \Pi^+$ , either  $[P]$  or  $(P)$ .*

For example, if  $P$  consists of a single plane, either all the triangles in  $P$  admit tilings, or all of the triangles in the interiors of the 2-cells in  $P$  do not admit tilings.

## 4.2 Vertex arrangements of triangles

We now discuss vertex arrangements by an arbitrary abstract *scalene* triangle in a little more detail. Let  $\mathsf{T}$  be an abstract scalene triangle, with vertices labeled  $\alpha, \beta, \gamma$ , reading clockwise. Let the edges opposite these vertices be labeled  $A, B, C$ . Let an overscore denote reflection. Thus, reading clockwise, the angles of  $\overline{\mathsf{T}}$  are  $\overline{\gamma}, \overline{\beta}, \overline{\alpha}$  and the edges are  $\overline{C}, \overline{B}, \overline{A}$ . For convenience, for all  $x \in \{\alpha, \beta, \gamma, A, B, C\}$  let  $\overline{x} = x$ , and for all  $X, Y \in \{A, B, C, \overline{A}, \overline{B}, \overline{C}\}$  we say  $X, Y$  are *compatible* if and only if  $X = \overline{Y}$  or  $X = Y$ .

For each  $r, s, t \in \mathbb{N}$ , let  $\mathcal{V}(rst)$  be the set of all vertex arrangements admitted by the abstract  $\mathsf{T}$ , in which there are a total of  $r$  copies of  $\alpha, \overline{\alpha}$ , a total of  $s$  copies of  $\beta, \overline{\beta}$  and a total of  $t$  copies of  $\gamma, \overline{\gamma}$  meeting at the central vertex. Naturally, we assume not all of  $r, s, t$  are zero; on the other hand, since we are considering *abstract* vertex arrangements,  $\mathcal{V}(200) \neq \emptyset$ , even though no (convex) geometric triangle could admit such a vertex configuration. In any case, as an example,  $|\mathcal{V}(224)| = 7$ ; one arrangement is shown in the middle of in Figure 9.

For any set  $P \subset \Pi$ , write

$$\mathcal{V}(P) := \bigcup_{\pi_{rst} \in P} \mathcal{V}(rst)$$

For any *geometric* triangle  $\mathsf{T} \subset \mathbb{H}^2, \mathbb{E}^2, \mathbb{S}^2$ , taking  $P \subset \Pi$  to be the planes on which the point  $\mathsf{T} \in \mathcal{O}$  lies, we write

$$\mathcal{V}(\mathsf{T}) := \mathcal{V}(P)$$

for the set of *abstract* vertex arrangements admitted by  $\mathsf{T}$ .

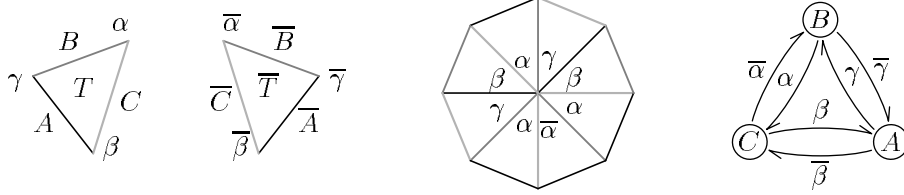


Figure 9: At left, the abstract triangle  $\mathsf{T}$  and its reflection  $\overline{\mathsf{T}}$ ; in the middle, the vertex arrangement  $\alpha\overline{\alpha}\beta\overline{\beta}\gamma\overline{\gamma}$  in  $\mathcal{V}(224)$ ; at right  $\Gamma_{\mathsf{T}}$ .

Let  $\Gamma_{\mathsf{T}}$  be the directed graph at right in Figure 9, with edges denoted  $\alpha, \beta, \gamma, \overline{\alpha}, \overline{\beta}, \overline{\gamma}$ , and vertices denoted  $A, B, C$ ; an edge runs from a vertex  $X$  to a vertex  $Y$  iff the corresponding angle lies between edges  $X, Y$  or  $\overline{X}, \overline{Y}$ . For a cycle  $l$  in  $\Gamma_{\mathsf{T}}$ , let  $l_{\alpha}, l_{\beta}, l_{\gamma}$  be the total number of times  $l$  runs along  $\alpha$  or  $\overline{\alpha}$ ,  $\beta$  or  $\overline{\beta}$ ,  $\gamma$  or  $\overline{\gamma}$ , respectively. Then let  $\Gamma_{rst}$  be the set of all cycles in  $\Gamma_{\mathsf{T}}$  with  $l_{\alpha} = r, l_{\beta} = s, l_{\gamma} = t$ . From the definitions:

**Lemma 4.6** *The cycles in  $\Gamma_{rst}$  precisely correspond with the vertex arrangements of  $\mathcal{V}(rst)$ ; the sequence of labels of the edges of a given cycle in  $\Gamma_{rst}$  precisely correspond to the sequence of labels of the vertices of a given vertex arrangement in  $\mathcal{V}(rst)$ , reading counterclockwise.*

Consequently:

**Lemma 4.7**  $\mathcal{V}(rst) \neq \emptyset$  if and only if  $r \equiv s \equiv t \pmod{2}$ ,  $r, s, t \geq 0$  and  $r + s + t \neq 0$ .

**Proof** Let  $v$  be any vertex arrangement whatsoever;  $v$  corresponds to a cycle  $l$  in  $\Gamma_{\mathsf{T}}$ . We must have that  $l_{\alpha}, l_{\beta}, l_{\gamma}$  each have the same parity as the winding number of  $l$  about the center face of the graph. Consequently if  $\mathcal{V}(rst)$  is not empty, then  $r \equiv s \equiv t \pmod{2}$ . Conversely, suppose  $r \equiv s \equiv t \pmod{2}$ , all non-negative and not all zero. It is not hard to obtain a cycle  $l$  with  $l_{\alpha} = r, l_{\beta} = s, l_{\gamma} = t$   $\square$

**Corollary 4.8** *Let  $P$  be any finite set of planes in  $\Pi$  and let  $Q$  be any set of planes  $\pi_{rst}$  in  $\Pi$  such that  $r, s, t$  are not all even and are not all odd; then  $[P]$  if and only if  $[P \cup Q]$  and  $(P)$  if and only if  $(P \cup Q)$ .*

That is,  $\mathcal{V}(Q) = \emptyset$  and  $\mathcal{V}(P \cup Q) = \mathcal{V}(P)$ ; the equations contributed by planes in  $Q$  contribute nothing to the combinatorics.

**Lemma 4.9** *Let  $\mathsf{T}$  be any triangle and let  $P \subset \Pi^+$  be maximal with  $\mathsf{T} \in P$ . Then  $\mathcal{V}(\mathsf{T}) \supset \mathcal{V}(P)$ , with equality holding if  $\mathsf{T}$  is scalene (i.e. if  $P \subset \Pi$ ).*

**Proof** Suppose  $\mathsf{T} \in \pi_{rst}$ . Then the angles  $\alpha, \beta, \gamma$  of  $\mathsf{T}$  satisfy  $r\alpha + s\beta + t\gamma = 2\pi$ . Choose any abstract vertex arrangement in  $\mathcal{V}(rst)$ . There is no geometric or combinatorial restriction to forming this arrangement, geometrically, with copies of  $\mathsf{T}$ . Hence  $\mathcal{V}(\mathsf{T}) \supset \mathcal{V}_P$ .

Conversely, suppose that  $\mathsf{T}$  is scalene and consider any vertex arrangement  $v$  with  $r, s, t$  copies of  $\alpha, \beta, \gamma$  meeting at the central vertex. Then by definition,  $v \in \mathcal{V}(rst)$  and  $\mathsf{T} \in \pi_{rst}$ . Hence  $\mathcal{V}(\mathsf{T}) = \mathcal{V}(P)$  for scalene  $\mathsf{T}$ .  $\square$

This means, essentially, that all triangles in a given  $k$ -cell of  $\mathcal{T}$  admit configurations of precisely the same combinatorial structures. That is:

Given a triangle  $T$ , we may regard any tiling  $\tau$  by  $T$  as a complex with labeled edges. Let  $T_1, T_2$  be triangles, and let  $\tau_1, \tau_2$  be tilings by  $T_1, T_2$ . We say that  $\tau_1, \tau_2$  are **homeomorphic** iff there is a labeling of the edges of  $T_1, T_2$  such that  $\tau_1$  and  $\tau_2$  are topologically homeomorphic as labeled complexes. This next lemma is key:

**Lemma 4.10** *Let  $P \subset \Pi^+$  and let  $T \in \mathring{\cap}P$ ,  $T' \in \cap P$ . For any tiling  $\tau$  by  $T$ , there is a homeomorphic tiling by  $T'$ .*

**Proof** Ignoring the geometry, we may consider  $\tau$  as a  $\mathcal{V}(T)$ -complex; since  $\mathcal{V}(T) \subset \mathcal{V}(T')$ , we have that  $\tau$  is a  $\mathcal{V}(T')$ -complex, and charting the geometry induced by  $T'$ , we have a tiling  $\tau'$  by  $T'$  that is homeomorphic to  $\tau$ .  $\square$

**Corollary 4.11** *Let finite  $P \subset \Pi^+$ . If any  $T \in \mathring{\cap}P$  admits a tiling, every  $T \in \cap P$  does; if any  $T \in \cap P$  does not admit a tiling, no  $T \in \mathring{\cap}P$  does.*

And Theorem 4.5 is a corollary of this.

## 5 Regular productions describing triangles

We now give an alphabet  $\mathcal{A}_T$  and a regular language  $\mathcal{L}_T$  suited for describing the combinatorics of tilings by triangles. The letters of the alphabet correspond to the ways two or three copies of  $T$  can meet at a point. The words of the language corresponds to ways these arrangements can be strung along a polygonal curve (Figure 10).

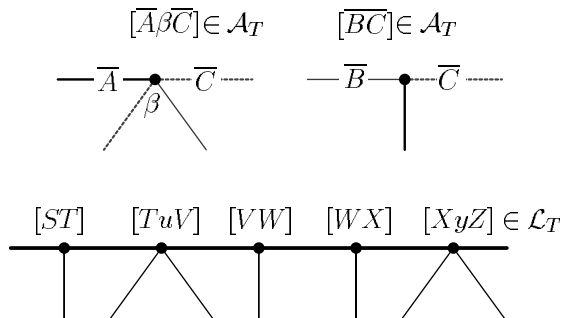


Figure 10: Our alphabet  $\mathcal{A}_T$  and language  $\mathcal{L}_T$

The letters in  $\mathcal{A}_T$  are each made of two or three symbols; they are of the form  $[XyZ]$  or  $[XZ]$  where  $X, Z \in \{A, B, C, \overline{A}, \overline{B}, \overline{C}\}$  and  $y \in \{\alpha, \beta, \gamma, \overline{\alpha}, \overline{\beta}, \overline{\gamma}\}$ . Not every string of symbols forms a letter though: the two or three triangles must actually fit together. In particular it is not hard to see that there are 12 letters made of two symbols and 24 letters made of three (these correspond to the paths of length 2 or 3 in the graph  $\Gamma_T$  of Figure 9, that is, to the ways of arranging two or three triangles around a vertex). In Figure 10, the letters  $[\overline{A}\beta\overline{C}]$ ,  $[\overline{B}\overline{C}]$  are illustrated.

The language can be described as paths in a digraph with six vertices denoted  $\{A, B, C, \overline{A}, \overline{B}, \overline{C}\}$ . The edges are labeled in  $\mathcal{A}$ : an edge  $[X..Y]$  runs from the vertex  $X$  to the vertex  $Y$ . That is, one letter may be followed by a second if the symbols on the end of the first matches the symbol at the beginning of the second. Hence  $[\overline{A}\beta\overline{C}][\overline{C}\alpha\overline{B}][\overline{B}\overline{C}] \in \mathcal{L}_T$ , for example. We will use some shorthand and abbreviate our words, unambiguously: for example, we will write  $\overline{A}\beta\overline{C}\alpha\overline{B}\overline{C}$  for  $[\overline{A}\beta\overline{C}][\overline{C}\alpha\overline{B}][\overline{B}\overline{C}]$ .

Note, as illustrated at the bottom of Figure 10 that each word in  $\mathcal{L}_T$  corresponds to an (abstract) strip of triangles running alongside a polygonal arc, so that two or three triangles meet at each vertex in the interior of the curve (and only one triangle meets the vertices on the

end of the curve). The Roman symbols in the abbreviated word exactly correspond with the sequence of edge labels along the curve, and the Greek symbols correspond with the vertex label at the central triangle at the vertices at which three triangles meet. We can make this more formal by explicitly describing a map from words to abstract strips, but this seems unnecessary.

From this point on, we will let  $\mathcal{A} := \mathcal{A}_\top$  and  $\mathcal{L} := \mathcal{L}_\top$ .

Let  $\mathcal{V}$  be any set of vertex arrangements of the abstract triangle. We will define a regular production system  $(\mathcal{A}, \mathcal{L}, \mathcal{R}_\mathcal{V})$  with respect to  $\mathcal{V}$  such that if there is an orbit in this system, there is a 2-complex, made of triangles, such that every vertex arrangement in the complex is a copy of one of the arrangements in  $\mathcal{V}$ .

We will only be using the arrangements in  $\mathcal{V}$  made of at least six triangles however; this makes our construction somewhat simpler. But it should be clear that there is no fundamental obstruction to using regular productions far more generally.

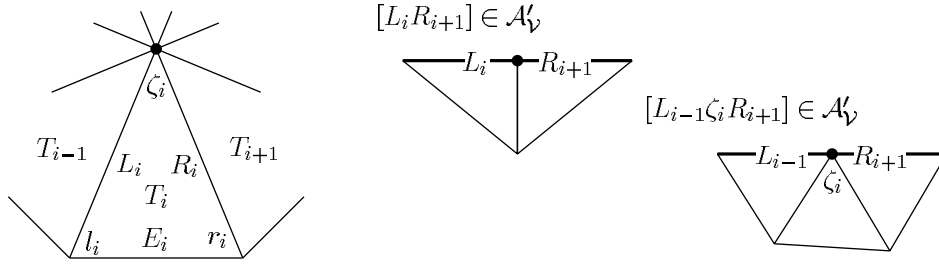


Figure 11: Our alphabet  $\mathcal{A}'_\mathcal{V}$

Let  $n_v$  be the number of triangles in a given vertex arrangement  $v$  and as illustrated at left in Figure 11, index these  $n_v$  triangles counterclockwise about the center of  $v$ . In the  $i$ th copy of  $T$  in  $v$ , let  $E_i, R_i, L_i \in \{A, B, C, \overline{A}, \overline{B}, \overline{C}\}$  be the labels of the outside edge, the “right” edge and the “left” edge of the triangle, and let  $\zeta_i, r_i, l_i \in \{\alpha, \beta, \gamma, \overline{\alpha}, \overline{\beta}, \overline{\gamma}\}$  be the labels of the central, “left” and “right” vertices of the triangle. We’ll consider these indices  $i$  modulo  $n_v$ .

As illustrated at top in Figure 12 now define a set  $\mathcal{R}_\mathcal{V}$  of rules: for each  $v \in \mathcal{V}$  with  $n_v \geq 6$ , and each cyclic permutation of the indices of the triangles in  $v$ , we set:

$$\begin{aligned} [L_1 R_2] &\mapsto E_{n-1} \dots E_4 r_3 X \\ [L_n \zeta_1 R_2] &\mapsto E_{n-2} \dots E_4 r_3 X \end{aligned}$$

where  $X \in \{\overline{R_3}, L_3\}$ . Note that  $\overline{R_3}, L_3$  are the only possibilities for the indicated edge label if the triangles are to fit together (the rightmost triangle is must be obtained from  $T_3$  either by rotating by  $\pi$  or reflecting across  $E_3$ ).

Typically, there will be a tremendous number of rules in  $\mathcal{R}'_\mathcal{V}$  (because of the size of the set  $\mathcal{V}$ ). For example, for the small case  $\mathcal{V}_{224}$  there are 224 rules; for  $\mathcal{V}_{333}$ , there are 528. But as we will soon see, we will be only making use of relatively tiny subsets of these.

These rules precisely define which strips can be fit together as indicated at the bottom of Figure 12:

A strip described by word  $\sigma \in \mathcal{L}_\mathcal{V}$  can be fitted above a strip described by word  $\omega \in \mathcal{L}_\mathcal{V}$  so that all the newly surrounded vertices on  $\omega$  are among the vertex arrangements of  $\mathcal{V}$  if and only if  $\omega \mapsto \sigma$ .

However, note that there may well be letters in  $\mathcal{A}$  to which no rule applies; moreover many of the rules may be “dead-ends” and lead to words to which in turn no rule may be applied. But worst of all, typically, one may apply a given rule to a given letter depending on what rules are applied to neighboring letters, which in turn depend on the rules applied to their neighbors and so on. So on the one hand, the tremendous number of rules in a typical  $\mathcal{R}_\mathcal{V}$  suggests there will be enough flexibility to apply *some* rules or another to each word and therefore there should



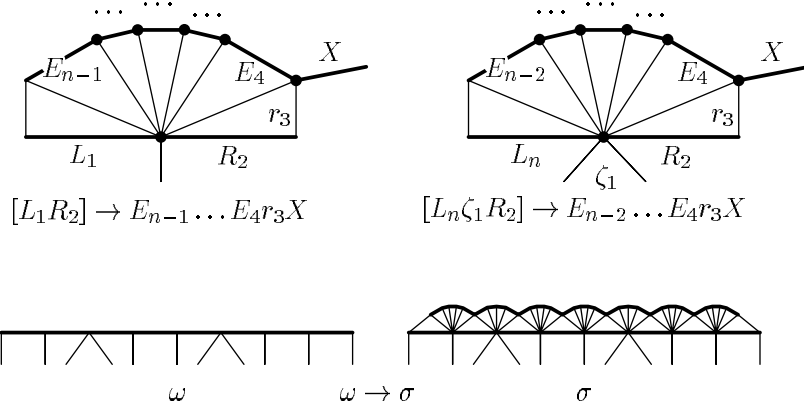


Figure 12: Our rules  $\mathcal{R}_\mathcal{V}$  and the relation  $\mapsto$

be orbits in the system  $(\mathcal{A}, \mathcal{L}, \mathcal{R}_\mathcal{V})$ . On the other hand, these systems are tricky enough that getting a precise hold on these orbits can be difficult.

The following is essentially tautological:

**Lemma 5.1** *Let  $T$  be a triangle and let  $\mathcal{V}$  be a collection of abstract vertex arrangements. Consider the regular production system  $(\mathcal{A}, \mathcal{L}, \mathcal{R}_\mathcal{V})$ . Then there exists a  $\mathcal{V}$ -complex if there is an orbit in  $\mathcal{L}^\infty$  under  $\mathcal{R}_\mathcal{V}$ .*

**Proof** Suppose there is an orbit in  $\mathcal{L}^\infty$  under  $\mathcal{R}_\mathcal{V}$ . But a word  $\omega$  in  $\mathcal{L}^\infty$  precisely corresponds to an infinite strip of triangles following a polygonal arc. And if  $\omega \mapsto \sigma$  the corresponding strips fit together perfectly, and the vertices along which they join are copies of the arrangements in  $\mathcal{V}$ . The theorem follows immediately.  $\square$

Now all that remains is to actually discuss which triangles give rise to vertex arrangements  $\mathcal{V}$  that in turn allow orbits in  $(\mathcal{A}, \mathcal{L}, \mathcal{R}_\mathcal{V})$ . But before going on, note as an aside:

**Lemma 5.2** *Suppose  $T$  is a (geometric) triangle in  $\mathbb{H}^2, \mathbb{E}^2$  and let  $\mathcal{V}$  be a set of vertex arrangements of  $T$  such that there is a periodic orbit in  $(\mathcal{A}, \mathcal{L}, \mathcal{R}_\mathcal{V})$ . Then in fact there exists a tiling of  $\mathbb{H}^2, \mathbb{E}^2$  by  $T$  that has an infinite cyclic symmetry.*

**Proof** Note that this is sensitive to the precise use of indexing in the orbit. Suppose there is a periodic orbit  $\{\omega^i\}$  with period  $n$ , and let  $T$  be the corresponding tiling. Then in  $T$ , consider the vertex arrangement corresponding to  $\omega_0^1$ ; the isometry taking  $\omega_0^1$  to  $\omega_0^{1+n}$  will leave  $T$  invariant and will be fixed-point free.  $\square$

In particular, in this paper and in [15], we exclusively use cuts and lifts to establish orbits. Thus, each time we establish the existence of orbits, we construct periodic orbits as well (Lemma 2.8) and for every triangle for which we succeed in constructing a tiling, we construct a tiling with an infinite cyclic symmetry. No triangle for which we produce a tiling here is strongly aperiodic [12],[13]. On the other hand, almost every triangle that tiles has area that is not in  $\mathbb{Q}\pi$  and so cannot tile with a co-compact symmetry. Almost every triangle that admits a tiling is weakly aperiodic.

## 6 Tilings by triangles

We pause to illustrate these techniques with a well-known theorem:

## 6.1 The Poincaré triangle theorem

**Theorem 6.1 (Poincaré)** *Let  $P = \{\pi_{(2p)00}, \pi_{0(2q)0}, \pi_{00(2r)}\}$ ,  $p, q, r \geq 2$ . Then  $[P]$ .*

In fact we only prove here  $[P]$  for  $p, q, r \geq 3$ , simply because our productions have been optimized for triangles meeting at least six-to-a-vertex. But the remaining cases are not at all difficult.

**Proof** Assume  $p, q, r \geq 3$ . Let  $P = \{\pi_{(2p)00}, \pi_{0(2q)0}, \pi_{00(2r)}\}$  and let  $T \in \cap P$ . Let  $\mathcal{V} = \mathcal{V}(P)$ . Then certainly the three vertex configurations  $(\alpha\bar{\alpha})^p, (\beta\bar{\beta})^q, (\gamma\bar{\gamma})^r$  lie in  $\mathcal{V}$  and the following rules lie in  $\mathcal{R}_{\mathcal{V}}$ :

$(\alpha\bar{\alpha})^p$	$(\beta\bar{\beta})^q$	$(\gamma\bar{\gamma})^r$
$[\overline{BB}] \mapsto (\overline{AA})^{i+1}\overline{\beta C}$	$[\overline{CC}] \mapsto (\overline{BB})^{j+1}\overline{\gamma A}$	$[\overline{AA}] \mapsto (\overline{CC})^{k+1}\overline{\alpha B}$
$[\overline{CC}] \mapsto (\overline{AA})^{i+1}\overline{\gamma B}$	$[\overline{AA}] \mapsto (\overline{BB})^{j+1}\overline{\alpha C}$	$[\overline{BB}] \mapsto (\overline{CC})^{k+1}\overline{\alpha A}$
$[\overline{B\bar{\alpha}C}] \mapsto A(\overline{AA})^i\overline{\gamma B}$	$[\overline{C\bar{\beta}A}] \mapsto B(\overline{BB})^j\overline{\alpha C}$	$[\overline{A\bar{\gamma}B}] \mapsto C(\overline{CC})^k\overline{\beta A}$
$[\overline{C\bar{\alpha}B}] \mapsto \overline{A}(\overline{AA})^i\overline{\beta C}$	$[\overline{A\bar{\beta}C}] \mapsto \overline{B}(\overline{BB})^j\overline{\gamma A}$	$[\overline{B\bar{\gamma}A}] \mapsto \overline{C}(\overline{CC})^k\overline{\alpha B}$

where  $i = p - 3$ ,  $j = q - 3$ ,  $k = r - 3$ ; recall our convention for abbreviating the words in  $\mathcal{L}$ . (For example,  $(\overline{AA})^2\overline{\beta C} = \overline{AA}\overline{AA}\overline{\beta C} = [\overline{AA}][\overline{AA}][\overline{AA}][\overline{A\beta C}]$ .)

Then taking  $\mathcal{A}^-$  to be the twelve letters on the left of each rule, taking  $\mathcal{L}^- = \mathcal{L}|_{\mathcal{A}^-}$  (that is,  $\mathcal{L}$  restricted to the letters of  $\mathcal{A}^-$ ) and  $\mathcal{R}^-$  to be specified by the twelve rules above, one can easily check that  $(\mathcal{A}^-, \mathcal{L}^-, \mathcal{R}^-)$  forms a cut of  $(\mathcal{A}, \mathcal{L}, \mathcal{R}_{\mathcal{V}})$ . Consequently, there is a orbit in  $\mathcal{L}^\infty$  under  $\mathcal{R}$ , there is a  $\mathcal{V}$ -complex, and finally,  $T$  admits a tiling.  $\square$

Indeed, this is precisely the core of Poincaré's own construction [18].

## 6.2 On planes in $\cup\Pi$

This Theorem completely settles the question of whether or not a given triangle, lying on exactly one plane of  $\Pi$ , admits a tiling. That is, we now have given necessary and sufficient conditions for admitting a tiling, for a measure-one set of triangles in  $\cup\Pi$ .

**Theorem 6.2** *Let  $P = \{\pi_{rst}\}$ . If  $r \equiv s \equiv t \pmod{2}$  and either  $r, s, t \geq 2$  or  $r = s = t = 1$ , then  $[P]$ ; otherwise  $(P)$ .*

A slightly weaker—but perhaps clearer—form of this theorem is: *Suppose a triangle  $T$  with vertex angles  $\alpha, \beta, \gamma$  satisfies exactly one equation of the form  $r\alpha + s\beta + t\gamma = 2\pi$ ,  $r, s, t \in \{0, 1, \dots\}$ . Then  $T$  admits a tiling if and only if  $r \equiv s \equiv t \pmod{2}$ , and either  $r, s, t \geq 2$  or  $r = s = t = 1$ .*

**Proof** Let  $T$  have vertex angles  $\alpha, \beta, \gamma$  satisfying  $r\alpha + s\beta + t\gamma = 2\pi$ ,  $r, s, t \in \{0, 1, \dots\}$ . Let  $P = \{\pi_{rst}\}$ . We first assume that the vertex angles satisfy only one such equation; that  $T \in \overset{\circ}{\cap} P$ . We will establish sufficient conditions for  $(P)$ :

First, if any of  $r, s, t = 0$ , it is not possible to form a vertex arrangement at each of the vertices of the triangle and  $T$  admits no tiling. Second if it is not true that  $r \equiv s \equiv t \pmod{2}$ , the triangle admits no vertex arrangements, and thus no tilings, by Lemma 4.7.

Suppose that, say  $r = 1$  and  $s \geq 1$ . By examining the graph in Figure 9, we see that in any vertex arrangement with  $\alpha$ , the pair  $\overline{\beta\beta}$  must appear. But following the edge between this pair of triangles, we must have the corners  $\alpha\bar{\alpha}$ , which cannot be completed into a vertex arrangement and cannot appear within a tiling. Similarly, if  $\bar{\alpha}$  appears, we must have  $\gamma\bar{\gamma}$ , and see  $\bar{\alpha}\alpha$ . The other cases are the same: if one of  $r, s, t$  equals 1 and another is greater than 1, then  $T$  does not admit a tiling.

We now establish sufficient conditions for  $[P]$ : let  $T \in \cap P$ . If  $r = s = t = 1$ , copies of  $T$  may be used to form a tetrahedral tiling of  $\mathbb{S}^2$ . Assume  $r, s, t \geq 2$ , with  $r \equiv s \equiv t \pmod{2}$ ; we take  $r \geq s \geq t$ .

We'll simply list the rules in a suitable (but by no means canonical) choice of  $\mathcal{R}^-$ ; the alphabet  $\mathcal{A}^-$  will be given as the letters on the left of the rules, and of course  $\mathcal{L}^-$  just is  $\mathcal{L}_T$  restricted to  $\mathcal{A}^-$ . Each rule is of the form:  $[XY] \rightarrow X \dots Y y Y$  or  $[XxX] \rightarrow X \dots X x X$  and is written in the letters of  $\mathcal{A}^-$ . Consequently,  $\mathcal{A}^-$ ,  $\mathcal{L}^-$  and  $\mathcal{R}^-$  is a cut for  $\mathcal{A}_\nu, \mathcal{L}_\nu, \mathcal{R}_\nu$ , and so by Lemma 2.9 we are done:  $\mathbb{T}$  admits a tiling and  $[P]$ .

For the most active reader, we list, to the right of the rules, the vertex arrangements that produce each rule; these are given by listing the  $\zeta_i$ 's.

Finally, note that the words in  $\mathcal{L}^-$  are abbreviated, as we discuss above. Hence  $[A\bar{A}] \rightarrow (A\bar{A})^i A\bar{A}\bar{\alpha}\bar{A}$  should be read as  $[A\bar{A}] \rightarrow ([A\bar{A}][\bar{A}A])^i [A\bar{A}][\bar{A}\bar{\alpha}\bar{A}]$ .

*Claim:*  $[P]$  for  $r = 2 + 2i, s = t = 2, i \geq 0$ .

$$\begin{array}{ll} [A\bar{A}] \rightarrow (A\bar{A})^i A\bar{A}\bar{\alpha}\bar{A} & \gamma\bar{\gamma}\bar{\beta}\bar{\alpha}\alpha(\bar{\alpha}\alpha)^i\beta \\ [\bar{A}A] \rightarrow (\bar{A}A)^i \bar{A}A\alpha A & \bar{\beta}\beta\gamma\alpha\bar{\alpha}(\alpha\bar{\alpha})^i\bar{\gamma} \\ [A\alpha A] \rightarrow (A\bar{A})^i A\alpha A & \gamma\alpha\beta\gamma\alpha(\bar{\alpha}\alpha)^i\beta \\ [A\bar{\alpha}\bar{A}] \rightarrow (\bar{A}A)^i \bar{A}\bar{\alpha}\bar{A} & \bar{\beta}\bar{\alpha}\bar{\gamma}\bar{\beta}\bar{\alpha}(\alpha\bar{\alpha})^i\bar{\gamma} \end{array}$$

*Claim:*  $[P]$  for  $r = 2 + 2i, s = 2 + 2j, t = 2, i, j \geq 0$ .

$$\begin{array}{ll} [A\bar{A}] \rightarrow (A\bar{A})^i (\bar{B}B)^j A\bar{A}\bar{\alpha}\bar{A} & \gamma\bar{\gamma}\bar{\beta}\bar{\alpha}\alpha(\beta\bar{\beta})^j(\bar{\alpha}\alpha)^i\beta \\ [B\bar{A}] \rightarrow B(A\bar{A})^i (\bar{B}B)^j A\alpha A & \alpha\beta\gamma\alpha(\beta\bar{\beta})^j(\bar{\alpha}\alpha)^i\beta\gamma \\ [B\bar{B}] \rightarrow B(A\bar{A})^i (\bar{B}B)^j B\bar{\beta}\bar{B} & \alpha\bar{\alpha}\bar{\gamma}\bar{\beta}(\beta\bar{\beta})^j(\bar{\alpha}\alpha)^i\beta\gamma \\ [\bar{A}A] \rightarrow \bar{A}(A\bar{A})^i (\bar{B}B)^j A\alpha A & \bar{\beta}\bar{\beta}\gamma\alpha(\beta\bar{\beta})^j(\bar{\alpha}\alpha)^i\bar{\alpha}\bar{\gamma} \\ [\bar{A}B] \rightarrow \bar{A}(A\bar{A})^i (\bar{B}B)^j B\bar{\beta}\bar{B} & \bar{\beta}\bar{\alpha}\bar{\gamma}\bar{\beta}(\beta\bar{\beta})^j(\bar{\alpha}\alpha)^i\bar{\alpha}\bar{\gamma} \\ [\bar{B}B] \rightarrow (\bar{B}B)^j (A\bar{A})^i \bar{B}B\beta B & \bar{\gamma}\gamma\alpha\beta\bar{\beta}(\bar{\alpha}\alpha)^i(\beta\bar{\beta})^j\bar{\alpha} \\ [A\alpha A] \rightarrow (A\bar{A})^i (\bar{B}B)^j A\alpha A & \gamma\alpha\beta\gamma\alpha(\beta\bar{\beta})^j(\bar{\alpha}\alpha)^i\beta \\ [B\beta B] \rightarrow B(A\bar{A})^i (\bar{B}B)^j \beta B & \alpha\beta\gamma\alpha(\beta\bar{\beta})^j(\bar{\alpha}\alpha)^i\beta\gamma \\ [A\bar{\alpha}\bar{A}] \rightarrow \bar{A}(\bar{B}B)^j (A\bar{A})^i \bar{\alpha}\bar{A} & \bar{\beta}\bar{\alpha}\bar{\gamma}\bar{\beta}(\bar{\alpha}\alpha)^i(\beta\bar{\beta})^j\bar{\alpha}\bar{\gamma} \\ [\bar{B}\beta\bar{B}] \rightarrow (\bar{B}B)^j (A\bar{A})^i \bar{B}\beta\bar{B} & \bar{\gamma}\bar{\beta}\bar{\alpha}\bar{\gamma}\bar{\beta}(\bar{\alpha}\alpha)^i(\beta\bar{\beta})^j\bar{\alpha} \end{array}$$

*Claim:*  $[P]$  for  $r = 2 + 2i + \delta, s = 2 + 2j + \delta, t = 2 + 2k + \delta$ , with  $i = j = k = \delta = 0$  or  $i, j, k \geq 0, \delta = 1, 2$ .

$$\begin{array}{lll} [AC] \rightarrow A^\rho C\gamma C & \gamma\alpha\beta\gamma\alpha^\rho\beta & [A\bar{A}] \rightarrow A^\rho \bar{A}\bar{\alpha}\bar{A} & \gamma\bar{\gamma}\bar{\beta}\bar{\alpha}\alpha^\rho\beta \\ [BA] \rightarrow B^\rho A\alpha A & \alpha\beta\gamma\alpha\beta^\rho\gamma & [B\bar{B}] \rightarrow B^\rho \bar{B}\bar{\beta}\bar{B} & \alpha\bar{\alpha}\bar{\gamma}\bar{\beta}\beta^\rho\gamma \\ [CB] \rightarrow C^\rho B\beta B & \beta\gamma\alpha\beta\gamma^\rho\alpha & [C\bar{C}] \rightarrow C^\rho \bar{C}\bar{\gamma}\bar{C} & \beta\bar{\beta}\bar{\alpha}\bar{\gamma}\gamma^\rho\alpha \\ [\bar{A}A] \rightarrow \bar{A}^\rho A\alpha A & \bar{\beta}\beta\gamma\alpha\bar{\alpha}^\rho\bar{\gamma} & [\bar{A}B] \rightarrow \bar{A}^\rho \bar{B}\bar{\beta}\bar{B} & \bar{\beta}\bar{\alpha}\bar{\gamma}\bar{\beta}\bar{\alpha}^\rho\bar{\gamma} \\ [\bar{B}B] \rightarrow \bar{B}^\rho B\beta B & \bar{\gamma}\gamma\alpha\beta\bar{\beta}^\rho\bar{\alpha} & [\bar{B}C] \rightarrow \bar{B}^\rho \bar{C}\bar{\gamma}\bar{C} & \bar{\gamma}\bar{\beta}\bar{\alpha}\bar{\gamma}\bar{\beta}^\rho\bar{\alpha} \\ [\bar{C}C] \rightarrow \bar{C}^\rho C\gamma C & \bar{\alpha}\alpha\beta\gamma\bar{\gamma}^\rho\bar{\beta} & [\bar{C}A] \rightarrow \bar{C}^\rho \bar{A}\bar{\alpha}\bar{A} & \bar{\alpha}\bar{\gamma}\bar{\beta}\bar{\alpha}\bar{\gamma}\bar{\gamma}^\rho\bar{\beta} \\ [A\alpha A] \rightarrow A^\rho \alpha A & \gamma\alpha\beta\gamma\alpha^\rho\beta & [B\beta B] \rightarrow B^\rho \beta B & \alpha\beta\gamma\alpha\beta^\rho\gamma \\ [C\gamma C] \rightarrow C^\rho \gamma C & \beta\gamma\alpha\beta\gamma^\rho\alpha & [A\bar{\alpha}\bar{A}] \rightarrow \bar{A}^\rho \bar{\alpha}\bar{A} & \bar{\beta}\bar{\alpha}\bar{\gamma}\bar{\beta}\bar{\alpha}^\rho\bar{\gamma} \\ [\bar{B}\beta\bar{B}] \rightarrow \bar{B}^\rho \bar{\beta}\bar{B} & \bar{\gamma}\bar{\beta}\bar{\alpha}\bar{\gamma}\bar{\beta}^\rho\bar{\alpha} & [\bar{C}\bar{\gamma}\bar{C}] \rightarrow \bar{C}^\rho \bar{\gamma}\bar{C} & \bar{\alpha}\bar{\gamma}\bar{\beta}\bar{\alpha}\bar{\gamma}\bar{\gamma}^\rho\bar{\beta} \end{array}$$

where if  $\delta = 0$ ,  $x^\rho = x$  for all  $x$ ; and if  $\delta = 1, 2$ , taking  $l = 2 - \delta$ ,

$$\begin{array}{ll} A^\rho = A(CBA)^l C(\bar{C}C)^k B(\bar{B}B)^j A(\bar{A}A)^i & \alpha^\rho = (\alpha\bar{\alpha})^i \alpha (\beta\bar{\beta})^j \beta (\gamma\bar{\gamma})^k \gamma (\alpha\beta\gamma)^l \alpha \\ B^\rho = B(ACB)^l A(\bar{A}A)^i C(\bar{C}C)^k B(\bar{B}B)^j & \beta^\rho = (\beta\bar{\beta})^j \beta (\gamma\bar{\gamma})^k \gamma (\alpha\bar{\alpha})^i \alpha (\beta\gamma\alpha)^l \beta \\ C^\rho = C(BAC)^l B(\bar{B}B)^j A(\bar{A}A)^i C(\bar{C}C)^k & \gamma^\rho = (\gamma\bar{\gamma})^k \gamma (\alpha\bar{\alpha})^i \alpha (\beta\bar{\beta})^j \beta (\gamma\alpha\beta)^l \gamma \\ \bar{A}^\rho = \bar{A}(\bar{B}C\bar{A})^l \bar{B}(\bar{B}B)^j \bar{C}(\bar{C}C)^k \bar{A}(\bar{A}A)^i & \bar{\alpha}^\rho = (\bar{\alpha}\alpha)^i \bar{\alpha} (\bar{\gamma}\bar{\gamma})^k \bar{\gamma} (\bar{\beta}\bar{\beta})^j \bar{\beta} (\bar{\alpha}\bar{\gamma}\bar{\beta})^l \bar{\alpha} \\ \bar{B}^\rho = \bar{B}(\bar{C}A\bar{B})^l \bar{C}(\bar{C}C)^k \bar{A}(\bar{A}A)^i \bar{B}(\bar{B}B)^j & \bar{\beta}^\rho = (\bar{\beta}\beta)^j \bar{\beta} (\bar{\alpha}\alpha)^i \bar{\alpha} (\bar{\gamma}\bar{\gamma})^k \bar{\gamma} (\bar{\beta}\bar{\alpha}\bar{\gamma})^l \bar{\beta} \\ \bar{C}^\rho = \bar{C}(\bar{A}B\bar{C})^l \bar{A}(\bar{A}A)^i \bar{B}(\bar{B}B)^j \bar{C}(\bar{C}C)^k & \bar{\gamma}^\rho = (\bar{\gamma}\gamma)^k \bar{\gamma} (\bar{\beta}\beta)^j \bar{\beta} (\bar{\alpha}\alpha)^i \bar{\alpha} (\bar{\gamma}\bar{\beta}\bar{\alpha})^l \bar{\gamma} \end{array}$$

Essentially we are inserting cycles of the graph of Figure 9 into the vertices to expand out the rules as needed.  $\square$

## References

- [1] A.F. Beardon, *The geometry of discrete groups*, Springer, New York (1983).
- [2] R. Berger, *The undecidability of the domino problem*, *Memoirs Am. Math. Soc.* **66** (1966).
- [3] J. Block and S. Weinberger, *Aperiodic tilings, positive scalar curvature and amenability of spaces*, *J. Am. Math. Soc.* **5** (1992), 907-918.
- [4] R. Bowen and C.M. Series, *Markov maps associated with Fuchsian groups*, *Publ. IHES* **50** (1979), 153-170.
- [5] J. W. Cannon, *The combinatorial structure of cocompact discrete hyperbolic groups*, *Geom. Dedicata* **16** (1984), 123-148.
- [6] J.W. Cannon, W.J. Floyd and W.R. Parry, *Crystal Growth, Biological Cell Growth, and Geometry*, *Pattern Formation in Biology, Vision and Dynamics*, A. Carbone, M. Gromov and P. Prusinkiewicz, eds., World Scientific (2000), 65-80.
- [7] L. Danzer, personal communication.
- [8] L. Danzer and N. Dolbilin, *Delone graphs; some species and local rules*, *The mathematics of long-range aperiodic order*, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. **489** (1997), Kluwer Acad. Publ., Dordrecht, 1997, 85-114.
- [9] H.L. Davies, *Packings of spherical triangles and tetrahedra*, *Proc. of the Colloq. on Convexity Copenhagen 1965*. Copenhagen (1967), 43-51.
- [10] D. Epstein and C. Petronio, *An exposition of Poincaré's Polyhedron Theorem*, *L'Enseignement Mathématique* **40** (1994), 113-170.
- [11] D. Epstein et al., *Word processing in groups*, Jones and Bartlett (1992).
- [12] C. Goodman-Strauss, *Open questions in tilings*, notes available at [comp.uark.edu/~cgstrauss/papers](http://comp.uark.edu/~cgstrauss/papers)
- [13] C. Goodman-Strauss, *A strongly aperiodic set of tiles in the hyperbolic plane*, *Inv. Math.* **159** (2005), 119-132.
- [14] C. Goodman-Strauss, *Drawing triangle tilings in the hyperbolic plane*, a *Mathematica* notebook available at [comp.uark.edu/~cgstrauss/papers](http://comp.uark.edu/~cgstrauss/papers)
- [15] C. Goodman-Strauss, *Further triangle tilings*, notes available at [comp.uark.edu/~cgstrauss/papers](http://comp.uark.edu/~cgstrauss/papers)
- [16] H. Hopf, *Zum Clifford-Kleinschen raumproblem*, *Mat. Annalen* **95** (1926), 313-339.
- [17] W. Kolakoski, *Problem 5304*, *Amer. Math. Monthly* **73** (1966), 681-682.
- [18] W. Magnus, *Noneuclidean tessellations and their groups*. Pure and Applied Mathematics, Vol. 61. Academic Press, New York-London (1974).
- [19] Ā. Marek, personal communication.
- [20] G.A. Margulis and S. Mozes, *Aperiodic tilings of the hyperbolic plane by convex polygons*, *Isr. J. Math.* **107** (1998), 319-325.
- [21] M.L. Minsky, *Computation and infinite machines*, Prentice-Hall, Englewood Cliffs, N.J. (1967).
- [22] B. Maskit, *On Poincaré's theorem for fundamental polygons*, *Adv. in Math.* **7** (1971), 219-230.
- [23] P. Prusinkiewicz and A. Lindenmeyer, *The Algorithmic Beauty of Plants*, Springer-Verlag (1990).
- [24] M. Queffelec, *Substitution dynamical systems— spectral analysis*, *Lecture Notes in Mathematics* **1294**, Springer-Verlag, New York (1987).
- [25] J.G. Ratcliffe, *Foundations of Hyperbolic Manifolds*, Springer-Verlag, New York (1994).
- [26] R.M. Robinson, *Undecidability and nonperiodicity of tilings in the plane*, *Inv. Math.* **12** (1971), 177-209.
- [27] R.M. Robinson, *Undecidable tiling problems in the hyperbolic plane*, *Inv. Math.* **44** (1978), 259-264.
- [28] W. Thurston, *Groups, tilings and finite state automata: Summer 1989 AMS colloquim lectures*.