A hierarchical strongly aperiodic set of tiles in the hyperbolic plane

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Abstract We give a new construction of strongly aperiodic set of tiles in $\mathbb{H}^2$, exhibiting a kind of hierarchical structure, simplifying the central framework of Margenstern's proof that the Domino problem is undecidable in the hyperbolic plane [13].

Ludwig Danzer once asked whether, in the hyperbolic plane, where there are no similarities, there could be any notion of hierarchical tiling—an idea which plays a great role in many constructions of aperiodic sets of tiles in the Euclidean plane [1, 2, 4, 5, 6, 15, 17, 18]. It is an honor to dedicate this paper, which exposes a way to look at this question, to Herr Prof. Danzer in his 80th year.

In 1966, R. Berger proved that the Domino Problem— whether a given set of tiles admits a tiling— is undecidable in the Euclidean plane, hanging his proof on the construction of an aperiodic set of tiles [2]. This first set was quite complex, with over 20,000 tiles; Berger himself reduced this to 104 [3] and in 1971, R. Robinson streamlined Berger's proof of the undecidability of the Domino Problem, working off of an aperiodic set of just six tiles [18]. Both Berger's and Robinson's constructions used, in a very strong way, the hierarchical nature of their underlying aperiodic sets of tiles.

In 1977, Robinson considered, but was unable to settle, the undecidability of the Domino Problem in the hyperbolic plane [19]. M. Margenstern recently gave a proof that the Domino Problem is undecidable in the hyperbolic plane [13]; despite the lack of scale invariance in this setting, he found a way to adapt and extend the Berger-Robinson construction. (J. Kari has independently given a completely different and highly original proof [11].)

Though it is difficult to discern—and Margenstern does not mention—the more than 18,000 tiles underlying his construction are a strongly aperiodic[1] set

[1]Over time, it became clear that when considering tilings outside of the Euclidean plane (in higher dimensions, or in curved spaces) one might distinguish between *weakly* aperiodic and *strongly* aperiodic sets of tiles [16].

Weakly aperiodic sets of tiles admit only tilings without a co-compact symmetry, i.e. without a compact fundamental domain. In the hyperbolic plane, this is an almost trivial property, enjoyed, for example, by the tiles in the n-fold horocyclic tiling described below.

Strongly aperiodic sets of tiles, in contrast, admit only tilings with no period whatsoever, tilings on which there is no infinite cyclic action. In the Euclidean plane, the two properties

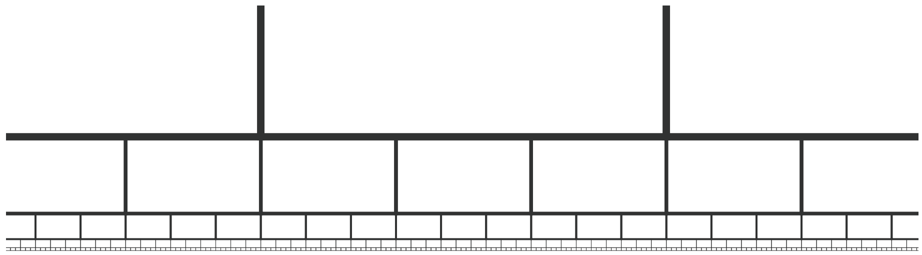


Figure 1: A 3-fold horocyclic tiling

of tiles, admitting only tilings with the kind of hierarchical structure we describe in Section 1.

Here, we attempt to distill the essence of Margenstern's complex construction, extracting his key idea in a much simplified form.

The construction in this paper will seem familiar to those acquainted with Robinson's classic proof; in some sense, we present only a trivial variation. But we remind the reader that this construction eluded many for a long while. In [13] Margenstern lights the way, and here we hope we smooth the path.

Readers familiar with Berger's or Robinson's proof of the undecidability of the Domino Problem in the Euclidean plane will have little trouble using the strongly aperiodic set of tiles in this paper to prove the undecidability of the Domino Problem in the hyperbolic plane.

1 The underlying idea

There is a standard template for presenting an aperiodic set of tiles: we describe the non-periodic structures we wish our tiles to form. We give the tiles themselves; we show the tiles can form these structures and admit tilings, and that they can only form these non-periodic structures and so are themselves aperiodic.

In Figure 1, we show what we will call the "n-fold horocyclic tiling", depicted in the upper-half plane model of the hyperbolic plane. The tiles are arranged hierarchically, each sitting above n other tiles.[2]

These tiles, we note, are not at all rectangular in the hyperbolic plane; though the vertical lines are straight geodesics, horizontal lines in the picture

imply one another (Theorem 3.7.1 of [9]), but in general this is not so (for example [20]). The first known strongly aperiodic set of tiles in the hyperbolic plane [7] was based on Kari's interesting aperiodic Wang tiles, based on sequences of Sturmian sequences [10].

[2]This generalizes quite nicely: we have horocyclic layers of tiles each of which can be viewed as a sequence of letters. A symbolic substitution system on letters relates one layer to the next, and an orbit in this system describes a tiling. Such a tiling has an infinite cyclic symmetry if and only if the orbit is periodic. Several authors have used this idea, in one form or another, less or more explicitly, to construct a variety of interesting tilings in the hyperbolic plane [8, 12, 14, 21].)

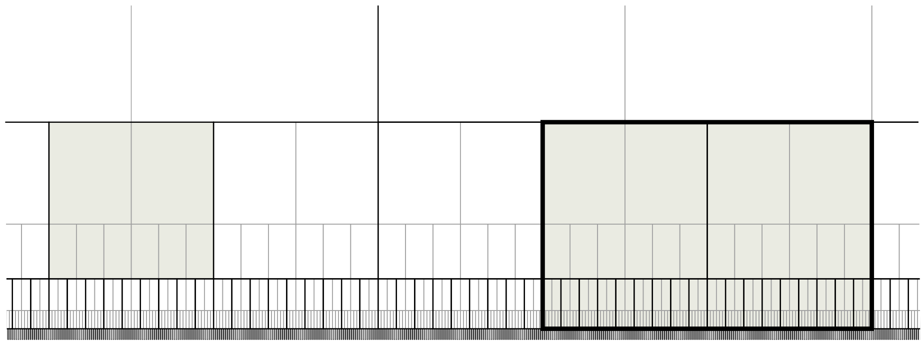


Figure 2: A hierarchy of (3^{2^k})-fold horocyclic tilings is shown in the "distorted upper-half plane model" of $\mathbb{H}^2$: the dark lines show a 9-fold tiling, one tile of which is highlighted at left. The bold line outlines a tile in a 81-fold tiling.

are horocycles in $\mathbb{H}^2$; the bottom edge of each tile bulges outward (is convex) and is quite a bit longer than the top edge, which bends inwards (is concave).

Allowing rotations of our tiles makes no difference— they can only fit together properly into horocyclic layers and any tiling by these tiles is locally congruent to the tiling in the figure. The tiles do admit uncountably many tilings— all of which look exactly the same underneath any given horocycle, and countably many of which have an infinite cyclic symmetry consisting of translations leaving some vertical geodesic invariant.

In the upper-half plane model, it is difficult to make proper illustrations of much of an n-fold horocyclic tiling; we will distort our images by the map $(x, y) \to (x, y^c)$ where $c < 1$ is some constant. This preserves the upper half-plane, but makes the widths of successive rows a bit more uniform.

Now the key observation is that *the tiles in a n-fold horocyclic tiling can be combined to form tiles in an n^2-fold horocyclic tiling.* In Figure 2 we show a 3-fold tiling, overlaid by a 3^2-fold tiling, and one tile in a 3^4-fold tiling. Continuing in this way, we can overlay an infinite family of (3^{2^k})-fold tilings, formed by rectangles $2^k \cdot 3^{2^k}$ times as wide (at their base), and 2^k times as tall as our initial 3-fold tiles.

A family of (n^{2^k})-fold horocyclic tilings overlaying one another is strongly non-periodic and no orientation preserving isometry can leave the family as whole invariant: Any individual (n^{2^k})-fold tiling can, at most, remain invariant only by a hyperbolic translation shifting vertically by some multiple of $n^{2^k}d$ where d is the distance between consecutive horocycles. But then, of course, no shift could leave all the tilings invariant in the family, and the structure as a whole is thus strongly non-periodic. A set of tiles that can only form tilings with this structure is therefore strongly aperiodic. We repeat this argument in Lemma 2

2 The tiles

As is often the case, we build up the construction, modifying the tiles in some set $\mathcal{S}$ to produce a set $\mathcal{S}'$ so that every tiling admitted by tiles in $\mathcal{S}'$ can be locally decomposed into a tiling by tiles in $\mathcal{S}$.

Let $n \equiv 1 \mod 4$, $n \geq 5$, and let T be a prototypical tile in the n-fold horocyclic tiling. Our tiles will be marked copies of T.

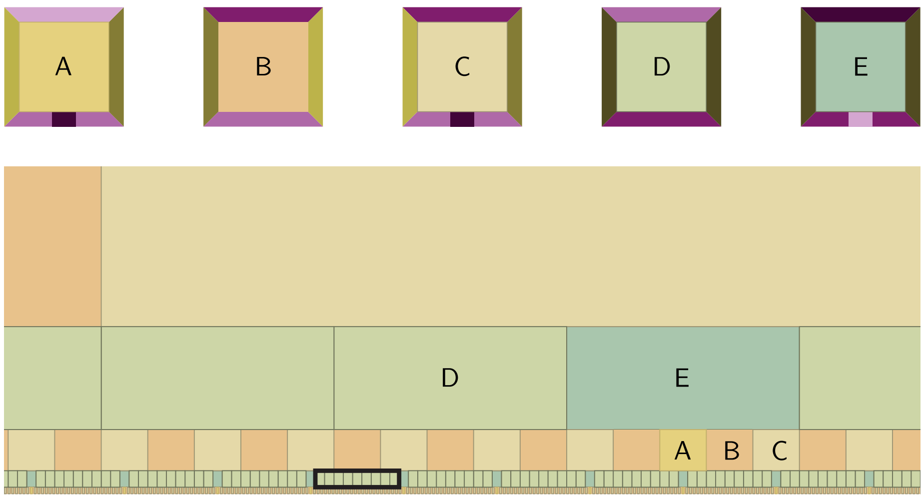


Figure 3: At top, five modified versions of the basic tile T; at bottom, a tiling by these tiles (the edge colors have been removed for clarity). Every tiling by these tiles is locally congruent to this one. The basic 0-level blocks $\mathcal{A}_0, \mathcal{B}_0$ and $\mathcal{C}_0$ are just the tiles A, B and C. The block $\mathcal{D}_0$ is a strip of $(2n-1)$ copies of D, outlined in black in the figure. For all $k \geq 0$, the block $\mathcal{E}_k$ is just the tile E.

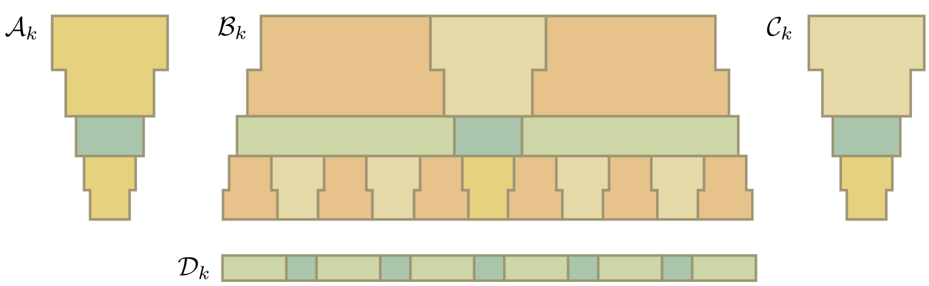


Figure 4: The k-level blocks $\mathcal{A}_k, \mathcal{B}_k, \mathcal{C}_k$ and $\mathcal{D}_k$ are inductively defined from $(k-1)$-level blocks.

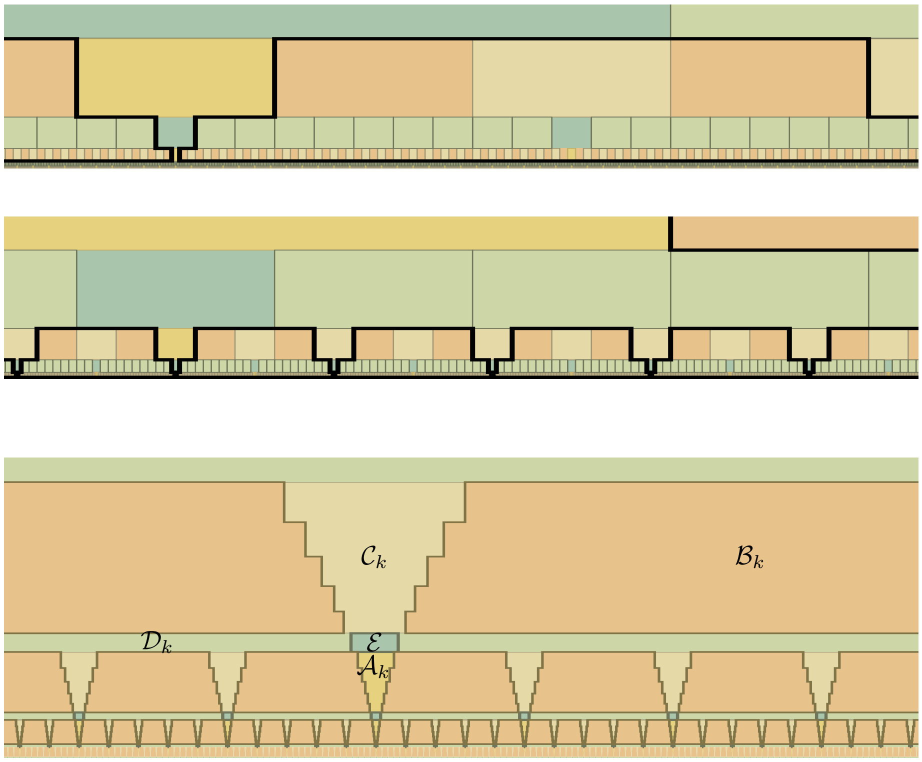


Figure 5: At top and middle, blocks $\mathcal{B}_1$ are outlined. Below, a schematic of a generic arrangement of k-level blocks.

2.1 The basic structure

We first describe five "basic" tiles, marked modifications of T, shown in Figure 3. These tiles shown at top of this figure can clearly only tile as shown below: rows of copies of A, B and C alternating with rows of consisting of $(2n-1)$ copies of D, then a copy of E; any A or C is directly above a E, which in turn is directly above an A.

We now describe a hierarchical structure we will try to force with additional matching rules. Inductively, for each $k = 0, 1, 2, \ldots$ we will define "k-level blocks" $\mathcal{A}_k, \mathcal{B}_k, \mathcal{C}_k, \mathcal{D}_k$ and $\mathcal{E}_k$, larger and larger configurations of tiles. First, $\mathcal{A}_0, \mathcal{B}_0$ and $\mathcal{C}_0$ are just the tiles A, B and C, and $\mathcal{D}_0$ is a horizontal strip of $(2n-1)$ copies of D. For all k, $\mathcal{E}_k$ consists of just a tile E. Note, as we go, that a E tile lies at the very center of any k-level block, $k > 0$.

As sketched in Figure 4, for $k \geq 1$ we define the blocks $\mathcal{A}_k$ ($\mathcal{C}_k$) to consist of a copy of $\mathcal{A}_{k-1}$ ($\mathcal{C}_{k-1}$), above an $\mathcal{E}$, above an $\mathcal{A}_{k-1}$. Inductively, then, $\mathcal{A}_k$ ($\mathcal{C}_k$) is a vertical strip with a copy of A (C) above $2^k - 1$ pairs E and A.

We define $\mathcal{B}_k$, $k \geq 1$ in three rows of smaller blocks: first, a row of three blocks, $\mathcal{B}_{k-1}$, $\mathcal{A}_{k-1}$ or $\mathcal{C}_{k-1}$, and then another $\mathcal{B}_{k-1}$; second, $\mathcal{D}_{k-1}$, $\mathcal{E}$ and $\mathcal{D}_{k-1}$; finally, $2\,n^{2^k}$ blocks $\mathcal{B}_{k-1}$, interleaved with $2\,n^{2^k}-2$ blocks $\mathcal{C}_{k-1}$ and one central $\mathcal{A}_{k-1}$.

Inductively, then $\mathcal{B}_k$ is $2^{k+1}-1$ tiles wide at its top and $2^{k+1}-1$ tiles tall; the rows of individual tiles are staggered outwards, something like a ziggurat. We'll call the block $\mathcal{E}$ the "core" of the block $\mathcal{B}_k$ and the central strips of blocks $\mathcal{A}_{k-1}$, $\mathcal{C}_{k-1}$ and $\mathcal{D}_{k-1}$, together with this core, the "spine" of $\mathcal{B}_k$. Note that in effect, the blocks $\mathcal{B}_k$, separated by strips one tile wide, form an $(n^{2^{k+1}})$-fold horocyclic tiling.

Finally, we define the blocks $\mathcal{D}_k$, $k \geq 1$ to be a horizontal strip of $2n^{2^k}$ copies of $\mathcal{D}_{k-1}$ interleaved with $2\,n^{2^k}-1$ copies of $\mathcal{E}$; $\mathcal{D}_k$ fits snugly beneath $\mathcal{B}_k$. Thus $\mathcal{D}_k$ consists of $2^k\,n^{2^{k+1}-2}$ copies of $\mathcal{D}_0$, interleaved with $2^k\,n^{2^{k+1}-2}-1$ copies of E.

A k-level block tiling is a partition of a tiling by $\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}$ and E into k-level blocks; a hierarchy of block tilings is a sequence of k-level block tilings, $\{\mathcal{T}_k\}$, $k = 0, 1, \ldots$ such that the k-level blocks of $\mathcal{T}_k$ partition the blocks of $\mathcal{T}_{k+1}$ as described in the inductive construction above. The proof of the first lemma is by construction:

Lemma 1 *For all $k = 0, 1, \ldots$ there exist k-level block tilings and there exist hierarchies of block tilings.*

We note:

Lemma 2 *No isometry leaves a hierarchy of block tilings invariant, with the sole possible exception of a single reflection about a vertical axis.*

Proof Any isometry preserving a family of horizontal horocycles can only be parabolic, fixing the point at infinity above the upper-half plane (i.e. a horizontal translation in the upper-half plane model); or must preserve a vertical geodesic (i.e. a hyperbolic transformation, a glide reflection or a reflection; in the upper-half plane model, these are a dilation through a point on the real axis, a glide dilation or a reflection). However these parabolic isometries cannot fix an n-fold horocyclic tiling: given horizontal shift of length x in the model, there is a row of tiles each of width, in the model, greater than x; this row will not be preserved. If a glide reflection leaves the hierarchy invariant, then repeating it twice, a hyperbolic transformation, will as well. No hyperbolic transformation can leave the hierarchy invariant, since for some k, the rows of k-level blocks will be taller than then length of this transformation. There can be a reflection across a vertical axis, but only one, since a pair would generate a parabolic isometry. (In fact, only one hierarchy will have even this much symmetry: such a reflection must preserve an infinite column of A and E tiles.) □

Since every D lies in a unique copy of $\mathcal{D}_0$, we will use as our set of basic tiles $\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathcal{D}_0$ and E.

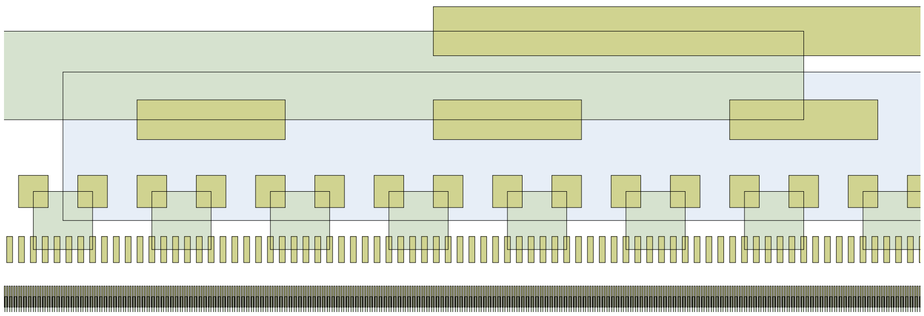
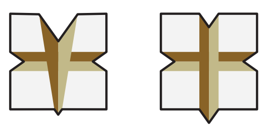


Figure 6: A schematic depiction of the regions bounded by nerve markings; different levels of the construction are indicated by different colors. At each level k, in a given finite region, the ratio of rectangles from row-to-row grows by $n^{2^{k+1}}$; the combinatorial structure exploits that this ratio is constantly $1 \mod 4$. The hierarchy of nerves of various levels plays a similar role here as do Margenstern's "threaded triangles" in [13]. (Indeed, these "rectangles" are increasingly "triangular" for larger and larger k.)

2.2 Nerve and mediating markings

We next give a second set of "nerve" and "mediating" markings, which we will overlay onto our five tiles $\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathcal{D}_0$ and E. Properly speaking, we should draw them as at left below, so the markings correctly match; for ease, though, we draw them instead as at right:



The basic idea is that each block $\mathcal{B}_k$ will have a rectangular "nerve" of markings that lie on the spines of its constituent $\mathcal{B}_{k-1}$ blocks. This nerve will define the structure of the block, and force markings on its own spine. (And in turn, these markings will be a part of some nerve of a larger $\mathcal{B}_{k+1}$ block.) In Figure 6 we sketch the structure of this hierarchy of nerves, and in Figure 7 we sketch how the markings we design will force $(k-1)$-level blocks to form k-level blocks.

All we must do is enumerate the markings we need, and the ways these markings can meet on a tile; we must then show that marked tiles can only form marked 1-level blocks, which can only form 2-level blocks etc. We establish that the marked tiles *do* admit tilings, but that the markings on any such tiling uniquely delineate a hierarchy of block tilings, and so any tiling by these tiles has no symmetry, and the tiles themselves must be *strongly aperiodic*. The most active readers will go further, and use this construction as the basis for a proof

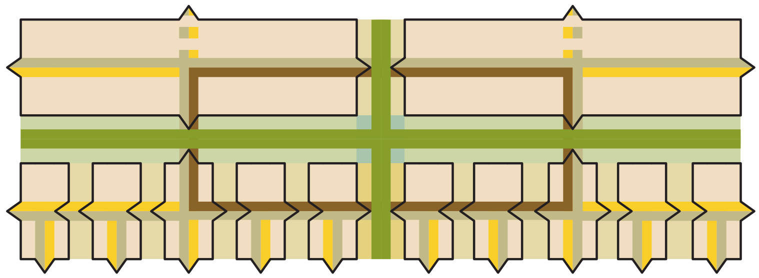


Figure 7: The combinatorics of the inductive construction. Markings on each "well-formed marked $\mathcal{B}_k$"; markings on the spine of the block's constituent $\mathcal{B}_{k-1}$ delineate a rectangular "nerve" of markings on a block $\mathcal{B}_k$, which in turn must bound a "spine" of markings, shown in green. (The actual number of blocks on the bottom row will grow rapidly with each stage of the inductive construction. The vertical mediating markings can vary: those leaving upwards from upper corners may or may not be present, and are shown as dashed. There are a great many dominated mediating markings, from lower level corners, between the $\mathcal{B}_{k-1}$ blocks, crossing the $\mathcal{A}_{k-1}, \mathcal{C}_{k-1}$ and, to a lesser extent, the $\mathcal{D}_{k-1}$ blocks. All other markings are as shown.)

that the Domino Problem is undecidable in the hyperbolic plane.

The "nerve" markings are shown as brown and beige in the following figures, and "mediating" markings are shown as yellow and beige. A marking may be "vertical" or "horizontal"; vertical markings are "left-sided" or "right-sided", depending on which side is colored beige, and "upward directed" or "downward directed" depending on which end has the outward pointing arrow. Similarly, horizontal markings are "upper" or "lower" and "left directed" or "right directed". We limit how markings may decorate a tile: In Figure 8 we decorate tiles B and E with "spines" and in Figure 9 we decorate tiles A, C, $\mathcal{D}_0$ and E with "crossings". In Figure 10 we decorate tiles A, C and E with "empty crossings". Recall that we may reflect left-to-right, but not top-to-bottom.

Let $\mathcal{S}$ be the set of 85 tiles we describe in the captions to the figures. (We have not made much effort to optimize this construction, and expect that fewer tiles could be used. Certainly, if we use non-standard matching rules, such as "tip-to-tip" rules, this set can be made considerably smaller.)

In the next few paragraphs, we define special configurations of tiles in $\mathcal{S}$: a "marked block" $\mathcal{X}_k$ is a block $\mathcal{X}_k$ with additional nerve and mediating markings. A "well-formed" marked block has additional properties, and, importantly, can be thought of as simply a larger version of one of the original tiles: For $k = 0$, the marked blocks are just the marked tiles themselves and these are considered well-formed.

Every marked $\mathcal{E}$ is just a marked E and is well-formed. In a well-formed marked $\mathcal{A}_k$ or $\mathcal{C}_k$, all of the constituent marked A, E and C tiles are vertically

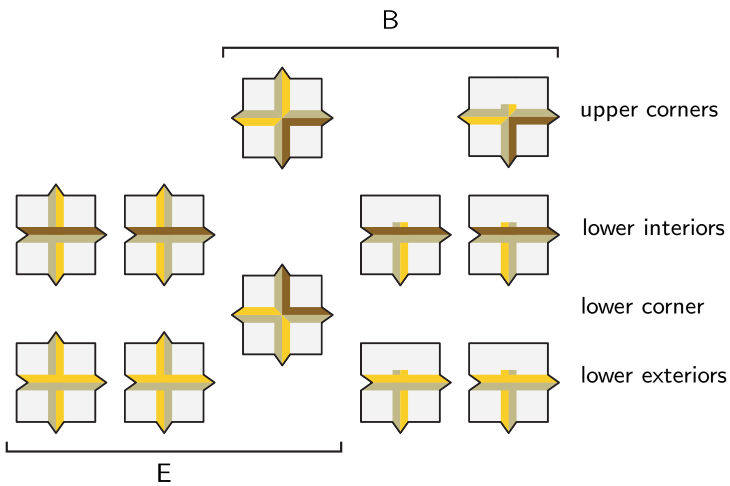


Figure 8: The "spines" in the third, fourth and fifth columns decorate B tiles; the spines of the first, second and third columns decorate E tiles. The spines in the middle column and/or first row are "corners"; the spines in the second row are "interior" and the spines in the last row are "exterior". The corners of the first row are "upper" and all remaining spines are "lower". All of these tiles are considered "left" though the horizontal markings on the interior and exterior spines are "right directed". Recall that we may reflect left-to-right but not top-to-bottom.

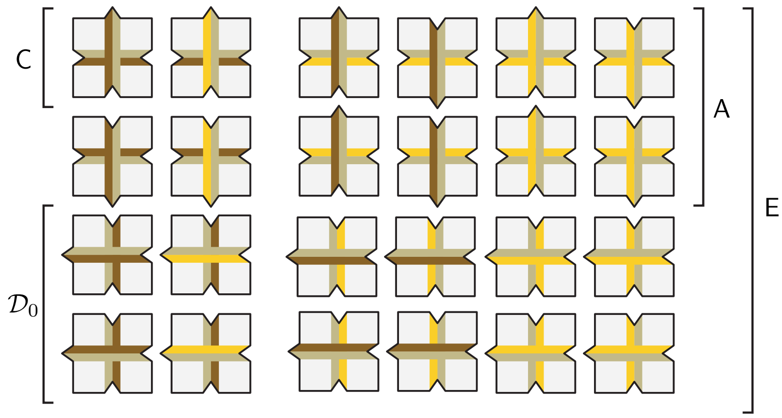


Figure 9: The "crossings" of the first and second rows are "vertically" dominated; the crossings of the third and fourthrows are "horizontally" dominated. The vertically dominated crossings decorate A and the horizontally dominated crossings decorate $\mathcal{D}_0$. The crossings of the first row, with an upper marking being dominated, decorate C, and the crossings of the all four rows decorate E. In the first two columns, nerve markings are dominated; note the direction of the dominating marking. In the final four columns, mediating markings are dominated, in all possible ways, with no restrictions. Recall we may reflect left-to-roght but not top-to-bottom.

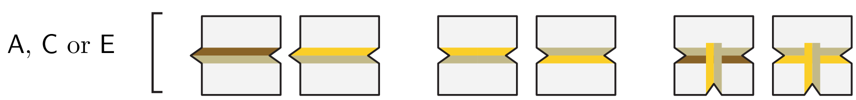


Figure 10: Finally, we decorate A, C and E tiles with "empty" crossings. The four empty crossings at left are considered to be horizontally dominated, and the two at right vertically dominated. All of these are considered to have an "empty" vertical marking.

dominated (possibly empty) crossings, and with the possible exception of the central E tile, the horizontal dominated marking is mediating. Consequently, in a well-formed marked $\mathcal{A}_k$ or $\mathcal{C}_k$, we only have three possibilities for the vertical markings:

(i) There is a single vertical marking continuing across all the constituent tiles and the block is much the same as one of the A or C tiles in the first two rows of Figure 9; if the dominated marking on the central E tile is a nerve marking, then the vertical marking is upward or downward directed, depending on the central nerve marking on the central E is upper or lower; or

(ii) The central E is one of the four at left in Figure 10: all of the vertical markings in all of the constituent tiles are empty and the block is much the same as one of the four A or C tiles at left in Figure 10; or

(iii) The central E has a dominated horizontal upper nerve marking; either this tile is the E fifth from left in Figure 10 or one of the tiles above it is the rightmost tile in Figure 10; above this point all of the vertical markings are empty, and there is a vertical mediating marking on all of the constituent tiles below. The block is much like one of the two rightmost A or C tiles in Figure 10.

In a well-formed marked $\mathcal{D}_k$, all of the constituent marked $\mathcal{D}_0$ and E tiles are horizontally dominated crossings, and consequently, all the tiles in a well-formed marked $\mathcal{D}_k$ have the same horizontal marking; the block is much the same as one of the D tiles on the bottom two rows of Figure 9. The dominated vertical marking of the central E tile may be a nerve or mediating marking, and if it is a nerve marking, the horizontal marking on all of the constituent tiles will be left or right directed, depending on whether the central vertical nerve marking is left- or right-sided. The vertical dominated marking on the remaining tiles is to be mediating or empty.

We define well-formed marked $\mathcal{B}_k$ blocks inductively (Figure 7): Recall, the decorated B are well-formed marked $\mathcal{B}_0$ tiles. For $k \geq 1$, first, the constituent $(k-1)$-level blocks are well-formed. Second, in a well-formed $\mathcal{B}_k$, the central $\mathcal{E}$ is marked with a spine; consequently, the markings of this $\mathcal{E}$ propagate outwards along the well-formed $\mathcal{D}_{k-1}$, $\mathcal{A}_{k-1}$ and $\mathcal{C}_{k-1}$ blocks that make up the the spine of a well-formed $\mathcal{B}_k$. Following the names of our spine tiles in Figure 8, we'll call a well-formed marked $\mathcal{B}_k$ a "corner", "interior" or "exterior", "upper", "lower", "left" or "right", depending on the markings of the central $\mathcal{E}$ and the spine.

Third, any other markings on the boundary of a well-formed $\mathcal{B}_k$ are outward directed mediating markings. Fourth, the spines of the constituent $\mathcal{B}_{k-1}$ are very specifically marked: the left upper $\mathcal{B}_{k-1}$ is an upper left corner; the upper right $\mathcal{B}_{k-1}$ is an right upper corner. The two $\mathcal{B}_{k-1}$ blocks directly beneath these, respectively, are left and right lower corners. All of the $\mathcal{B}_{k-1}$ blocks between these lower corners are lower interiors and the remaining $\mathcal{B}_{k-1}$ blocks are lower exteriors. The downward directed mediating markings on the lower interiors and lower exteriors are alternately left- and right-sided. (Recall that $n \equiv 1 \mod 4$; and so for all k, $(n^{2^k} - 1)/4$ is an integer.) On the bottom row, there are $(n^{2^k} - 1)/4$ pairs of left lower exterior blocks, a left lower corner, $(n^{2^k} - 1)/4$ pairs of left lower interior blocks, the spine, $(n^{2^k} - 1)/4$ pairs of right lower interior blocks, a right lower corner and then $(n^{2^k} - 1)/4$ pairs of right lower exterior blocks.) Finally, the the vertical markings in the remaining $\mathcal{C}_k$ blocks are all empty. Figure 7 summarizes these properties.

The following is implied by the above, but can be taken as an additional part of the definition of well-formed marked $\mathcal{B}_k$ blocks if we wish: First, all of the $\mathcal{C}_{k-1}$ blocks on the bottom row have empty vertical markings, as in (ii) above. Second, the central $\mathcal{A}_{k-1}$ or $\mathcal{C}_{k-1}$ on the top row is either as in (i) above, if the central $\mathcal{E}$ is as in the third column of Figure 8, or as in (iii), if the central $\mathcal{E}$ is as in one of the first two columns of this figure. Finally, the markings in the central $\mathcal{A}_{k-1}$ must be as in (i) above, with a downward directed vertical marking.[3]

The following lemma follows from our inductive construction; it should not be difficult, with the descriptions provided above, to verify that we have provided all of the tiles needed to carry this out:

Lemma 3 *There exist, for all $k \geq 0$, well-formed marked blocks $\mathcal{B}_k$.*

Consequently, since these blocks cover disks of arbitrarily large radius, by a standard argument (see Theorem 3.8.1, or the beginning of Chapter 11, in [9]):

Corollary 4 *The set $\mathcal{S}$ admits tilings of the hyperbolic plane.*

Taking note that each corner lies on a unique rectangular nerve, an inductive argument shows

Lemma 5 *In any tiling by $\mathcal{S}$, for any k, all well-formed marked $\mathcal{B}_k$ are disjoint.*

We next define "a well-formed k-level block tiling" to be a tiling by tiles in $\mathcal{S}$ such that every B tile lies in a (unique, by Lemma 5) well-formed marked $\mathcal{B}_k$ and such that contracting all tiles A, C and D to line segments and E to points produces an (n^{2^k})-fold horocyclic tiling. That is, in such a tiling, all the $\mathcal{B}_k$ blocks lie in well-defined rows and any given $\mathcal{B}_k$ block is directly aligned above n^{2^k} $\mathcal{B}_k$ blocks.

We now come to the main theorem:

[3] In essence, if the central $\mathcal{E}$ is as in the first two columns of Figure 8, a $\mathcal{B}_k$ is combinatorially much the same as one of the corners in the last two columns of Figure 8, and the vertical mediating marking terminates somewhere in the upper half of the central upper $\mathcal{A}_{k-1}$ or $\mathcal{C}_{k-1}$. Otherwise, this $\mathcal{B}_k$ is much the same as the central $\mathcal{E}$ itself.

Theorem 6 *Every tiling by $\mathcal{S}$ is a well-formed k-level block tiling, for all k.*

Proof The proof is by induction on k; trivially every tiling by $\mathcal{S}$ is a well-formed 0-level block tiling. Consider a well-formed k-level block tiling by tiles in $\mathcal{S}$; we will show it must be a well-formed $(k+1)$-level block tiling (Figure 7).

Any well-formed marked block $\mathcal{B}_k$ is upper or lower, and any upper (lower) well-formed marked $\mathcal{B}_k$ can only be horizontally adjacent to another upper (lower) well-formed marked $\mathcal{B}_k$. Moreover, directly beneath the center of an upper (lower) well-formed marked $\mathcal{B}_k$ can only be a lower (upper) well-formed marked $\mathcal{B}_k$. Consequently, the rows of well-formed marked $\mathcal{B}_k$ blocks are alternately upper and lower.

A left upper well-formed marked $\mathcal{B}_k$ can only be to the left of a right marked $\mathcal{B}_k$, and vice versa. Thus on the rows of upper well-formed marked $\mathcal{B}_k$, the blocks are alternately left and right. The vertical nerve markings on the spines of these ensure that directly beneath the center of any left (right) upper well-formed marked $\mathcal{B}_k$ there can only be a left (right) lower well-formed marked $\mathcal{B}_k$.

Thus, any upper corner must belong to a set of four: a pair of upper corners above a pair of lower corners, left above left, right above right. Is it possible that there are additional corners between the lower corners we've forced? Any additional corners would have to be lower, and thus direct nerve markings upwards; but there is nowhere available to absorb these and so no additional corners are possible. There can only be interior nerve markings, directed right from the left hand corner and leftwards from the right hand corner.

Thus every upper corner must belong to a rectangular nerve of corners and interior markings, surrounding a marked nerve tile E. Can there be lower corners that are not already accounted for? No because, again, there is no room for an upward directed marking to propagate. Consequently, all lower nerves are either exterior, or on one of the rectangular nerves just described.

What of the remaining tiles and blocks between these $\mathcal{B}_k$? Along the top of the nerve, between the left and right facing upper corner $\mathcal{B}_k$, there is a dominated horizontal nerve marking on a central E tile, and many dominated horizontal mediating markings on A, C and E tiles. Consequently, there is a well-formed marked $\mathcal{A}_k$ or $\mathcal{C}_k$ between these two blocks.

Similarly, underneath either of the upper corner blocks, there is a dominated vertical nerve marking on an E tile and dominated vertical mediating or empty markings on the remaining E and $\mathcal{D}_0$ tiles beneath that upper corner block. Thus, beneath either of the two upper corners there can only be a well-formed marked $\mathcal{D}_k$ block.

In the middle of the rectangular nerve is a E tile; on the nerve above and to either side of this central E is a dominated nerve marking and so this central E must be a marked spine of some kind, with outwards directed markings propagating to the nerve. Some vertical marking must propagate downwards from this central E, dominating horizontal mediating markings along the way, until meeting the bottom of the nerve on a E tile.

Now somewhere along the bottom of the nerve, just once, between two of the well-formed marked interior $\mathcal{B}_k$, there must be a dominated horizontal nerve marking, since those on the left are right directed and those on the right are left directed. However, nowhere but the central E is there a location at which such a downward directed marking can originate; consequently, beneath the central E is a well-formed marked $\mathcal{A}_k$, with some downward directed marking; all of the other tiles between the well-formed lower $\mathcal{B}_k$ either have empty vertical markings and dominated horizontal mediating markings, or are on the nerve (and are the marked E at left in Figure 10), or are mediating exterior markings on this same row as the nerve (and are marked E as at second-to-left in Figure 10).

In short, in any well-formed k-level block tiling, the blocks $\mathcal{B}_k$ lie in well-formed blocks $\mathcal{B}_{k+1}$. We now only need show these larger $\mathcal{B}_{k+1}$ blocks, in turn, are aligned correctly. But they must in fact lie in rows, since an upper corner $\mathcal{B}_k$ of one can only be horizontally adjacent to the upper corner of another. And one $\mathcal{B}_{k+1}$ lies cleanly above $n^{2^{k+1}}$ others: Fix a $\mathcal{B}_{k+1}$ and consider the row of $\mathcal{B}_{k+1}$ beneath it. In this lower row, the constituent upper corner $\mathcal{B}_k$ blocks are grouped together in pairs and we must only check that this grouping has the right parity. But the left (right) lower corner of the nerve of this upper $\mathcal{B}_{k+1}$ lie directly above the left (right) upper corner of a $\mathcal{B}_{k+1}$ block on the row below. There are an even number of corner $\mathcal{B}_k$ blocks to the left (right) of these, and so an integral number of $\mathcal{B}_{k+1}$ blocks. All together, we have shown that these well-formed marked $\mathcal{B}_{k+1}$ blocks lie in a well-formed $(k+1)$-level block tiling, and so we are done. □

Corollary 7 *In any tiling by tiles in $\mathcal{S}$, for every $k \geq 0$, every marked* B *lies in a unique well-formed marked $\mathcal{B}_k$.*

Hence any tiling by $\mathcal{S}$ admits a unique hierarchy of blocks, and so is not preserved by any infinite cyclic isometry. We have

Proposition 8 *The set $\mathcal{S}$ of 85 marked tiles are strongly aperiodic. In particular, any tiling by these tiles is a hierarchy of blocks.*

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