Matching rules and substitution tilings

By Chaim Goodman-Strauss[1]

Abstract

A substitution tiling is a certain globally defined hierarchical structure in a geometric space; we show that for any substitution tiling in $\mathbb{E}^{\mathsf{d}}$, $\mathsf{d} > 1$, subject to relatively mild conditions, one can construct local rules that force the desired global structure to emerge. As an immediate corollary, infinite collections of forced aperiodic tilings are constructed. The theorem covers all known examples of hierarchical aperiodic tilings.

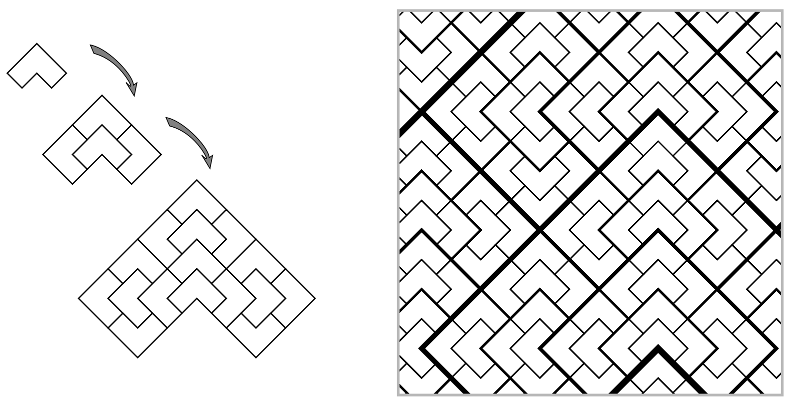

Figure 1: A substitution tiling

On the left in figure 1, L-shaped tiles are repeatedly "inflated and subdivided". (We define our terms more precisely in Section 1) As this process is iterated, larger and larger regions of the plane are tiled with L-tiles hierarchically arranged into larger and larger images of inflated and subdivided L-tiles, as at right in figure 1 (the thicker lines have been added to emphasize the hierarchy). We can then define a global structure– the "substitution tiling" induced by the inflation and division of the tiles.

But L-tiles can tile the plane in myriad ways. Is there a set of local conditions— "matching rules" — that, if satisfied everywhere, *force* the hierarchical structure of the substitution tiling to emerge? One can show that no such rules exist for *unmarked* L-tiles. However, we can find a set of *marked* L-tiles, and matching

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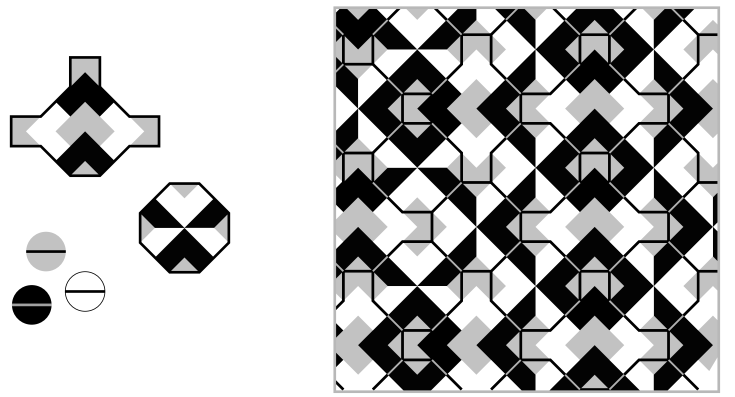

Figure 2: A matching rule tiling

rules that force the the original hierarchical structure to emerge. For example, in figure 2, we tile with the two marked tiles at upper left, and require that the colors "black", "grey" and "white" along their edges, as indicated by the three circles at lower left. At right we see a portion of a tiling satisfying these rules, which can clearly be recomposed into the original L-tiling.

As the hierarchical structure of figure 1 is precisely reproduced, we say the original substitution tiling has been "enforced" by these matching rules. (In [12] we show, by a different method than used here, that these marked tiles *must* recreate the hierarchical structure of figure 1.)

That is, this global structure can be recreated using only locally defined conditions.

Matching rules have been given for a variety of hierarchical tilings, beginning with R. Berger's landmark paper [1]. R. Robinson gave the first simple example [20], soon followed by R. Penrose's celebrated rhombs [18], much studied by J. Conway, eg. [13]. R. Amman [13], J. Socolar [24], L. Danzer [5] and others have constructed many beautiful examples. S. Mozes gave rules enforcing one special infinite class of substitution tiling [17], and C. Radin gave rules enforcing the Conway-Radin pinwheel tiling [19]. However, no truly general technique had emerged.

Our theorem includes all examples known to the author:

Theorem *Every substitution tiling of* $\mathbb{E}^{\mathsf{d}}$, $\mathsf{d} > 1$, *can be enforced with finite matching rules, subject to a mild condition:*

the tiles are required to admit a set of "hereditary edges" such that the substitution tiling is "sibling-edge-to-edge".

Equivalently, all substitution tilings satisfying our mild condition "admit local matching rules after decoration" (cf. [16]).

In effect the condition is simply that we require that the substitution breaks the edges of the parent into edges of the children and that siblings' edges coincide if they overlap at all. In [7], for each $\mathbb{E}^{\mathsf{d}}$, infinite families of substitution tilings satisfying the condition are presented. Although there is no ready example *not* satisfying the condition, the condition does not seem immediately implicit in the basic definition of a substitution.

We sharply require $\mathsf{d} > 1$: it is well known that no 1-dimensional substitution tiling can be enforced by matching rules, and our construction completely breaks down when $\mathsf{d} = 1$ (cf. Section 2.1.4).

Our theorem immediately gives an infinite collection of corollaries: any substitution tiling satisfying our technical condition yields a set of "aperiodic tile"— tiles that do tile the plane but admit no "periodic" tiling. (Our construction produces aperiodic tiles even if the original substitution tiling is periodic; this is because we specifically enforce the hierarchical structure of the tiling; this hierarchical structure is not invariant under any translation and hence is not periodic.)

Two general methods are known to produce aperiodic tiles— tiles that admit no periodic tiling: construct matching rules enforcing substitution tilings as in the theorem above; or construct matching rules enforcing a "quasiperiodic" tiling- tilings derived as slices through higher dimensional lattices, eg. [3, 22]. Le T.T.Q. has recently given a theorem similar to ours, for certain classes of quasiperiodic tilings [16].

Exactly two other classes of forced aperiodic tilings are known. The first was found by P. Schmitt and altered by J.H. Conway and L. Danzer [4]. These tilings are of $\mathbb{E}^3$ and are not isotropic. A new class emerged in 1996: J. Kari tiled the plane with tiles reminiscent of Wang's constructions of the early 1960's [14].

A fuller introduction to substitution tilings and many issues of technical or historical interest, as well as a detailed sketch of the proof, has been relegated to [7]. Further detailed examples in the style of the Appendix are available from the author [9]. Finally, much of the foundation for this and other papers is laid out in [6]

The proof is ultimately rather simple: given a substitution tiling, we select certain structures which, after being encoded into a matching rule tiling, if they appear at some level of the substitution hierarchy must appear at all levels. However, explicitely describing these structures and showing they can always be found is quite time-consuming.

In Section 1 we will establish the setting, defining basic terms such as "tile", "substitution" and "enforcement".

In Section 2 we will select vertices and edges, and carefully define our "labels"— names of elements in structures we draw from the substitution tiling. Our main structures will be "skeletons" and "vertex-wires".

In Section 3, we describe how to mark tiles with our labels and in Section 3.5 give matching rules.

In Section 4 we prove the matching rules and marked tiles enforce the original substitution tiling.

In the Appendix matching rules are produced for a specific substitution tiling.

1 Definitions

1.1 Tiles, matching rules, matching rule tilings

We take for our space and congruences, d-dimensional Euclidean space $\mathbb{E}^{\mathsf{d}}$, $1 < \mathsf{d} \in \mathbb{N}$. $\mathcal{G}$ will be the set of Euclidean isometries on $\mathbb{E}^{\mathsf{d}}$.

A **prototile** A is a d-dimensional compact set in $\mathbb{E}^{\mathsf{d}}$, perhaps marked with some combinatorial information; it is understood that any image of any point in a marked prototile is also marked (in [10] markings are placed on firm set-theoretic ground).

A **tile** is a congruent image BA, $\mathsf{B} \in \mathcal{G}$ of a prototile A.

A **tiling** $\cup \mathsf{B}_i\mathsf{A}_i$ of $X \subseteq \mathbb{E}^{\mathsf{d}}$ by prototiles $\mathcal{T}$ satisfies: $X = \cup \mathsf{B}_i\mathsf{A}_i$, and for $i \neq j$, $\mathsf{B}_i\mathsf{A}_i$ and $\mathsf{B}_j\mathsf{A}_j$ have disjoint interiors.

If we do not specify X, we assume $X = \mathbb{E}^{\mathsf{d}}$. We will use the symbols $=,\subset$, etc. in a special way for tilings: Given two tilings τ, τ', when we write, e.g., $\tau \subset \tau'$ we not only mean that the points in τ are a subset of the points in τ' but that the tiles in τ are among the tiles in τ'.

A set of tilings of X by prototiles $\mathcal{T}$, perhaps with further restrictions, is a **species** of tilings.

"Configurations" are often defined as a tiling of some compact X, but for later convenience we take: A **configuration** is a subset $\mathsf{D} \subset \cup \mathsf{B}_i\mathsf{A}_i$ of (marked) points in a tiling.

A **finite set $\mathcal{M}$ of matching rules** for a tiling of X with prototiles $\mathcal{T}$ is simply a set of restrictions on which tiles may be adjacent to each other, in which positions. (A full discussion of various formal frameworks for matching rules has been relegated to [10]). The simplest form of these restrictions is simply to require the tiles fit together— this is sufficient for our needs.

A tiling $\cup \mathsf{B}_i \mathsf{A}_i$ **satisfies** a set $\mathcal{M}$ of matching rules if and only pair of adjacent tiles in the tiling is permitted under the matching rules.

A **matching rule tiling** $(\mathcal{M}, \mathcal{T}')$, is the species of tilings of $\mathbb{E}^{\mathsf{d}}$, with prototiles $\mathcal{T}'$ that satisfy matching rules $\mathcal{M}$.

For example, in figure 2, we have two tiles in $\mathcal{T}'$ marked with colors Black, Gray, and White. Our set of matching rules simply requires that the colors match on the boundaries of neighboring tiles. The image on the right of figure 2 satisfies these matching rules.

1.2 An inflation σ and substitutions $\mathcal{S}$

A **inflation** will be an expanding linear map acting on $\mathbb{E}^{\mathsf{d}}$; that is, a linear map under which all distances increase. Let σ be a inflation on $\mathbb{E}^{\mathsf{d}}$, with center of dilation at an oriented and fixed origin.

We assume henceforth that $\mathsf{d} > 1$. We first inductively define a d-dimensional **polyhedron** to be a d-complex Λ such that: Λ is a connected compact d-manifold embedded in $\mathbb{E}^{\mathsf{d}}$, each k-cell (k-*facet*), $0 \leq k \leq \mathsf{d}$, of Λ is a polyhedron in the exterior of Λ that lies in a k-plane in $\mathbb{E}^{\mathsf{d}}$, and the boundary of the polyhedron is tiled by $(\mathsf{d}-1)$-facets; a two dimensional polyhedron is a polygon. We further require the set of facets of a polyhedron to be finite. The terms "vertex" and "edge" will have special technical meanings relative to this complex, described below in Section 1.4.

Let $\mathcal{T}$ be a finite set of marked polyhedra in $\mathbb{E}^{\mathsf{d}}$. With no loss of generality, we may assume the union of 1-facets of any polyhedron in $\mathcal{T}$ is connected. (The 1-facets will allow us to ensure a well-formed supertile (Section 1.3) has a well-defined orientation.)

We essentially just inflate each tile and subdivide the image into congruences of our prototiles to define **substitutions** $\sigma' : \mathcal{T} \to \{\cup \mathsf{C}_i \mathsf{B}_i\}$, a map from the prototiles to tilings such that

$\sigma'(\mathsf{A})$ is a tiling $\cup \mathsf{C}_i \mathsf{B}_i$ of $\sigma(\mathsf{A})$

and for each of the C_i, for each $n \in \mathbb{N}$, there is a $\mathsf{C}_i^{(n)} \in \mathcal{G}$ with $\sigma^n \mathsf{C}_i = \mathsf{C}_i^{(n)} \sigma$

The last condition ensures we may repeatedly substitute: for example, for $\mathsf{A} \in \mathcal{T}$ define $(\sigma')^n(\mathsf{A}) = (\sigma')^{n-1}(\sigma'(\mathsf{A})) = (\sigma')^{n-1}(\cup \mathsf{C}_i \mathsf{B}_i)) = \cup \mathsf{C}_i^{(n-1)}(\sigma')^{(n-1)}(\mathsf{B}_i)$, etc. We thus may recursively generate larger and larger **supertiles** $\sigma^n(\mathsf{A})$.

For simplicity we coalesce σ and σ' and refer to both the substitution and the inflation as σ. We hope the context will make the meaning clear. For example, we may need to refer to the image under σ^n of some structure X in a prototile A, relative to the image of the prototile itself. It seems sensible to leave the use of σ slightly ambiguous and refer to $\sigma^n(X) \subset \sigma^n(\mathsf{A})$, whether the substitution or the inflation has acted on X (at left in figure 4). In particular, σ almost always will act as a substitution on prototiles, tiles and supertiles; σ will almost always act as a inflation on other structures.

That the map is expanding ensures that all k dimensional facets, $0 < k \leq \mathsf{d}$ of every prototile are eventually subdivided, for each is bounded and falls into one of only finitely many congruence classes.

To ensure that every tile is really used, we require, given $\mathcal{T}$ and σ that for each $\mathsf{A} \in \mathcal{T}$ there exists $n \in \mathbb{N}, \mathsf{B} \in \mathcal{T}, \mathsf{C} \in \mathcal{G}$ such that $\mathsf{CA} \subset \sigma^n(\mathsf{B})$.

The **level** of the supertile $\sigma^n(\mathsf{A})$ is n. We also will call the congruent images $\mathsf{B}\sigma^n(\mathsf{A})$ of supertiles, supertiles as well, and usually will take $\sigma^n(\mathsf{A})$ to mean any congruent image of $\sigma^n(\mathsf{A})$.

Given a supertile $\sigma^k(\mathsf{A}) = \cup\sigma^{(k-1)}(\mathsf{C}_i(\mathsf{B}_i))$, the $\sigma^{(k-1)}(\mathsf{C}_i(\mathsf{B}_i))$ are **daughter** supertiles relative to their *parent* supertile $\sigma^k(\mathsf{A})$, and are **sibling** supertiles relative to each other.

When we refer to one supertile inside another, e.g. $\sigma^j(\mathsf{B}) \subset \sigma^n(\mathsf{A})$, it will be understood that $\sigma^j(\mathsf{B})$ is a **descendant** supertile of **ancestral** $\sigma^n(\mathsf{A})$

Note the requirement that a inflated copy of a prototile is congruent to the union of its offspring is very strong indeed, and at first glance seems not to be obeyed in some well known examples, such as the Penrose rhombs [13, 18], or fractal examples such as the tilings of Thurston and Kenyon [15, 26]. However, these examples can all be recomposed into a tiling with substitution satisfying our requirements; there is undoubtedly a simple sufficient condition (such as "edge-to-edge", with "finitely many congruence classes of tile") for when this can be done.

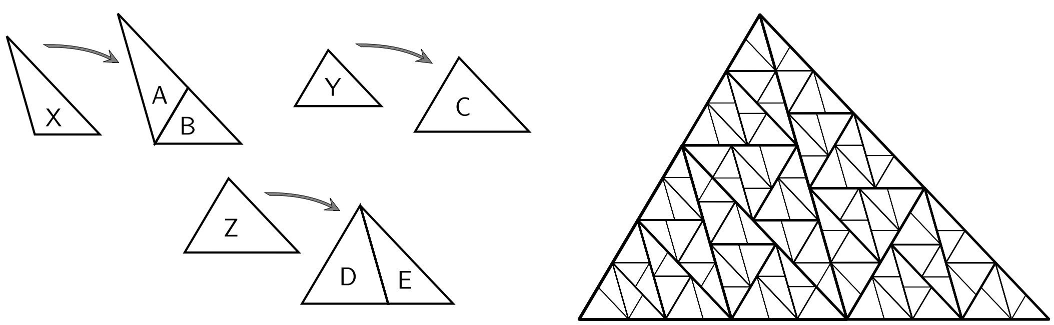

Figure 3: A substitution, with $\mathcal{T} = \{\mathsf{X}, \mathsf{Y}, \mathsf{Z}\}, \mathcal{S} = \{\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}, \mathcal{E}\}$ and a supertile $\sigma^9(\mathsf{Y})$, (reduced)

At left in figure 3 are three prototiles and their substitutions under σ. The C_i might be regarded as the motions needed to assemble such a diagram. On the right is an image of an 9-level supertile.

Let $\mathcal{S}$ be the disjoint union over T of the tiles $\mathsf{C}_i\mathsf{B}_i$. There is a natural projection from $\mathcal{S}$ to $\mathcal{T}$: each C_iB_i is mapped to B_i. Thus, a label in $\mathcal{S}$ gives a prototile's name in $\mathcal{T}$ but also provides additional information, the label of the prototile's parent in $\mathcal{T}$. (A **label** is simply a name in some defined class of names, such as $\mathcal{T}$ or $\mathcal{S}$). Often we will treat an element of $\mathcal{S}$ as an element of $\mathcal{T}$; consistently we will regard the elements of $\mathcal{S}$ as prototiles carrying the names of their parents in $\mathcal{T}$.

For $A \in \mathcal{T}$, the collection of possible predecessors (parents) of A is $A^- \subset \mathcal{T}$; the unique predecessor of $A \in \mathcal{S}$ is $A^- \in \mathcal{T}$; for either $A \in \mathcal{T}$ or $A \in \mathcal{S}$ the successors (daughters) of A are $A^+ \subset \mathcal{S}$. If for $A, B \in \mathcal{S}$ there is $C \in \mathcal{T}$ with $A, B \in C^+$, A and B are sibling supertiles. We will use other genealogical nomenclature as needed. (See the middle and right of figure 4)

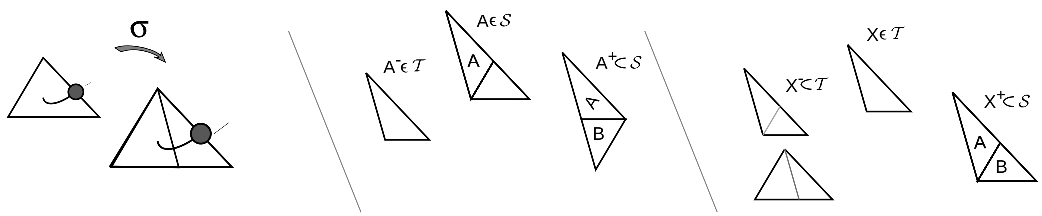

Figure 4: Conventions

Sometimes in designing substitution tilings, some proto-tiles serve as **place-holders** and are only subdivided after some finite number of inflations (as in figure 3). Note that because there are only finitely many prototiles and because our substitution is actually inflating our tiles, we must eventually subdivide our tiles, as well as any of their k-dimensional faces, $0 < k \leq d$.

A **substitution tiling** $(\mathcal{T}, \sigma, \mathcal{S})$ is the species of tilings $\cup B_iA_i$ by $\mathcal{T}$ such that any bounded configuration in any $\cup B_iA_i$ is congruent to some configuration in the interior of some supertile $\sigma^n(A)$, $A \in \mathcal{T}$, $n \in \mathbb{N}$.

Lemma 1.1 *Let $\cup B_iA_i$ be a tiling in $(\mathcal{T}, \sigma, \mathcal{S})$. Then for all B_jA_j, there is a nested sequence of supertiles $B_jA_j = \sigma^0(A_0) \subset \sigma^1(A_1) \subset \ldots \subset \sigma^n(A_n) \subset \ldots \subset \cup B_iA_i$ with each $A_n \in \mathcal{S}$, $A_n = A^-_{n-1}$.*

This lemma relates our intuition about what a substitution tiling should be— nested sequences of supertiles— to the actual definition above.

Note we are being casual about specifying that each $\sigma^n(A_n)$ is really a *congruent image* of a supertile.

Proof . The proof is relatively straightforward; a stronger version of this Lemma is given as Proposition 3.1.1 in [6]. □

We define an **infinite level supertile** to be the union of the supertiles in such a sequence. (This is defined in a more technically rigorous manner in [6]. In particular, there we note that every substitution tiling can be partitioned into infinite-level supertiles. Consequently, we call the union of edges between to infinite-level supertiles an **infinite-level fault line**).

We will map via $\lambda_{\mathcal{A}}$ the set of tiles BA in an infinite-level supertile $\mathcal{A}$ by $\lambda_{\mathcal{A}}(\mathsf{BA}) = \ldots \mathsf{A}_n \ldots \mathsf{A}_1 \mathsf{A}\bullet$ where each $\mathsf{A}_i \in S$, $A_i \in A^{+}_{i+1}$, $A_1 = A$, and $\sigma^i(A_i) \subset \sigma^{i+1}(A_{I+1})$. This object is used only twice in our proof, in Lemma 1.5 and in Section 2.2, but is in fact quite versatile [6].

A tiling $\cup \mathsf{B}_i \mathsf{A}_i$ in $(\mathcal{T}, \sigma, \mathcal{S})$ has **connected hierarchy** if and only if for every $x, y \in \cup \mathsf{B}_i \mathsf{A}_i$ there exists some supertile $\sigma^n(\mathsf{A})$ such that $x, y \subset \sigma^n(\mathsf{A}) \subset \cup \mathsf{B}_i \mathsf{A}_i$.

The tilings in $(\mathcal{T}, \sigma, \mathcal{S})$ with connected hierarchy capture the structure of the species:

(i) Every bounded configuration in any tiling in $(\mathcal{T}, \sigma, \mathcal{S})$ is congruent to a configuration in a tiling in $(\mathcal{T}, \sigma, \mathcal{S})$ with connected hierarchy ([6]).

(ii) In any $\mathcal{G}$-invarient probability measure on $(\mathcal{T}, \sigma, \mathcal{S})$, the set of tilings in $(\mathcal{T}, \sigma, \mathcal{S})$ with connected hiearchy has measure 1 ([17, 19]).

(iii) The tilings with connected hierachy are each covered (perhaps non-uniquely) by a single infinite-level supertile ([6]).

Lemma 1.2 *If a tiling has connected hierarchy then for each point x in the tiling, there exists an $N \in \mathbb{N}$ such that for all $n > N$ there exists an $\mathsf{A} \in \mathcal{T}$ with x in the interior of $\sigma^n(\mathsf{A})$.*

Proof This is straightforward. □

1.3 Enforcement

We now define "enforcement" of a substitution tiling by matching rules. An equivalent, more formal definition is given in [10].

A **labeling** λ of a substitution tiling $(\mathcal{T}, \sigma, \mathcal{S})$ is an algorithm for uniquely marking every tiling in $(\mathcal{T}, \sigma, \mathcal{S})$, such that the original markings of the prototiles $\mathcal{T}$ are unambiguously visible, and such that for any supertile $\sigma^n(\mathsf{A})$, there are only finitely many ways its images in the tilings in $(\mathcal{T}, \sigma, \mathcal{S})$ will be marked under λ.

For example, in figure 2, a labeling is given for the L-tilings; if one looks closely, one sees that each the images of each inflated L-tile are marked in one of only three ways.

Note that this definition is global: the labeling algorithm requires examining the entire infinite tiling, at least to label some points. For example, one may need to decide, for a given edge in a tiling, what is the highest level supertile with that edge on its boundary.

Once we have a labeling for $(\mathcal{T}, \sigma, \mathcal{S})$, let $\lambda(\sigma^n(\mathsf{A}))$ be the finite collection of labelings of the supertile $\sigma^n(\mathsf{A})$ and define a **well-formed supertile of level** n in $(\mathcal{M}, \mathcal{T}')$ to be a configuration X of tiles in $\mathcal{T}'$, satisfying $\mathcal{M}$, such that there exist $\mathsf{A} \in \mathcal{T}$, $\mathsf{B} \in \mathcal{G}$, and $\lambda' \in \lambda(\sigma^n(\mathsf{A}))$ such that all the markings of $\lambda'\sigma^n(\mathsf{A})$ coincide with the markings of the tiles in $\mathsf{B}X$.

For example in figure 2, the well formed supertiles are the pieces of the new tiles that lie in labeled inflated L-tiles.

A matching rule tiling $(\mathcal{M}, \mathcal{T}')$ **enforces** a substitution tiling $(\mathcal{T}, \sigma, \mathcal{S})$ if and only if one can define a labeling on $(\mathcal{T}, \sigma, \mathcal{S})$ such that for every $n \in \mathbb{N}$, every point in the interior of any tile in any tiling in $(\mathcal{M}, \mathcal{T}')$ lies in a unique well-formed supertile of level n.

A substitution tiling $(\mathcal{T}, \sigma, \mathcal{S})$ is **enforced by finite matching rules** if and only if there is a finite set of matching rules $\mathcal{M}$, and a finite set $\mathcal{T}'$ such that $(\mathcal{M}, \mathcal{T}')$ enforces $(\mathcal{T}, \sigma, \mathcal{S})$.

In effect, we have defined enforcement as being able to parse the matching rule tiling into supertiles. Our definition is effectively no stronger or weaker than other definitions known to the author.

1.4 Vertices and edges

We will use the terms "edges" and "vertices" in very technical senses. Note we strongly use $\mathsf{d} > 1$.

An **edge** of a prototile A in a substitution tiling $(\mathcal{T}, \sigma, \mathcal{S})$ is a $(\mathsf{d}-1)$-dimensional polyhedral subset e of some $(\mathsf{d}-1)$-facet of the polyhedron underlying the tile, such that e is the image, under the inflation σ^{-k}, for some $k \in \mathbb{N}$, of the union of $(\mathsf{d}-1)$-facets in the boundary of the supertile $\sigma^k(\mathsf{A})$. That is, edges arise by subdividing facets, and new edges can be derived by subdividing previously defined edges. Note facets are edges.

A **set of edges for a prototile** is a set of edges that cover the boundary of the prototile and have disjoint interiors. We will repeatedly redefine the elements of these sets of edges throughout the construction: in Section 2.1.1 we will begin with a set $\mathcal{E}$ edges that are among the facets of our prototiles; in Section 2.1.4 we will inflate and subdivide these to produce a class $\mathcal{E}'$; finally in Section 2.2.2, a third class $\mathcal{E}''$ will be derived from $\mathcal{E}'$.

An **edge of a tile** BA is the image under B of an edge of A; an **edge of a supertile** is the image of the edge of a tile under inflation σ^n; the **level** of an edge is n. Note that each point of an edge of a tile is coincident to one or more other edges, of neighboring tiles. Finally, an **edge of a tiling** is the image of the edge of a supertile in a tiling; note that each edge of a tile in a tiling belongs to exactly one edge of the tiling, and is coincident to a least one other edge of the tiling. (Our edges are more like "sides of edges", in the context of a tiling).

Once we have defined a set of edges of a prototile, we define the **vertices** of a prototile to be the points incident to 0-dimensional facets of the edges. Thus the vertices are discrete.

The **vertices of a tile** BA, $\mathsf{B} \in \mathcal{G}$, $\mathsf{A} \in \mathcal{T}$, are simply the images of the vertices of A under B. The vertices of a tiling are the points in the tiling that are vertices of tiles in the tiling; a vertex of a tiling may be coincident to a vertex of all or some, but at least one of the tiles incident to the vertex. The **endpoints** of an edge are the vertices of the edge.

A set of edges is **hereditary** if: for any edge e of any prototile A, $\sigma(e) \subset \sigma(\mathsf{A})$ is exactly tiled by edges of the tiles in A^+, and every k-facet, $0 \leq k \leq (\mathsf{d}-2)$ of $\sigma(e)$ is tiled by k-facets of tiles in $\sigma(A)$.

Similarly a set of vertices is **hereditary** if for every vertex v of prototile A, for each $B \in \mathsf{A}^+$ incident to $\sigma(v) \subset \sigma(\mathsf{A})$, there is a vertex of B coincident to $\sigma(v)$.

A substitution tiling, with edges defined, is **sibling edge-to-edge** if for each $\mathsf{B}, \mathsf{C} \in \mathsf{A}^+$, if any k-facet, $0 \leq k < \mathsf{d}-1$ of any edge e of B is incident to C in $\sigma(\mathsf{A})$, then this facet e is exactly coincident to some k-facet of some edge f of C.

A substitution tiling, with vertices defined, is **sibling vertex-to-vertex** if for each $\mathsf{B}, \mathsf{C} \in \mathsf{A}^+$, if a vertex v of B is incident to C in $\sigma(\mathsf{A})$, then v is exactly coincident to some vertex of C.

These conditions are indeed mild: one can always find edges for the images of the prototiles in the substitutions such that the tiling is sibling edge-to-edge. And one can always take as edges the $\mathsf{d}-1$ facets of the polyhedra underlying the prototiles; these will always be hereditary edges. An example of a tiling not satisfying the condition may not be hard to find, however, especially for $\mathsf{d} > 2$.

Still, for the time being it is unknown if the conditions are always satisfied.

Lemma 1.3 *If a set $\mathcal{E}$ of edges for a substitution tiling is hereditary, [the tiling is sibling-edge-to-edge], then the corresponding vertices $\mathcal{V}$ are hereditary, [the tiling is sibling-vertex-to-vertex]. When $\mathsf{d} = 2$, the converse holds as well.*

Proof This immediately follows from the definitions. □

However, the converse is probably not true when $\mathsf{d} > 2$: there are likely to be tilings that are sibling vertex-to-vertex that are not sibling edge-to-edge.

1.5 Epi-, Meso- and Endo- vertices

We give technical definitions of **endovertices**, **mesovertices** and **epivertices**. (This is used in Section 2.3).

First for each $\mathsf{A} \in \mathcal{T}$ let $\mathcal{E}(\mathsf{A})$ be $(\mathsf{d}-1)$-facets of the tiles in the interior of $\sigma(A)$. Note that the elements of $\mathcal{E}(\mathsf{A})$ are edges of the elements of A^+.

Second, by hypotheses, we can define hereditary vertices $\mathcal{V}(\mathsf{A})$ for each $\mathsf{A} \in \mathcal{S}$, such that the substitution tiling $\{\mathcal{T}, \sigma, \mathcal{S}\}$ is sibling-vertex-to-vertex. Note that the $\mathcal{V}(\mathsf{A})$ are a set of vertices if we take the $(\mathsf{d}-1)$-facets of A as edges for A.

Then we say:

$v \in \mathcal{V}(\mathsf{A})$ is an **endovertex** if and only if: both there is some (minimal) positive integer $\kappa(v)$ such that $\sigma^{\kappa(v)}(v)$ is incident to some edge in the interior of $\sigma^{\kappa(v)}(\mathsf{A})$ (and hence $\sigma^k(v)$ incident to some edge in the interior of $\sigma^k(\mathsf{A})$ for all $k \geq \kappa(v)$), and also v is incident to some $e \in \mathcal{E}(\mathsf{A}^-)$ in $\sigma(\mathsf{A}^-)$.

$v \in \mathcal{V}(\mathsf{A})$ is an **mesovertex** if and only if: v is incident to some $e \in \mathcal{E}(\mathsf{A}^-)$ in $\sigma(\mathsf{A}^-)$, but $\sigma^k(v)$ is not incident to any edge in the interior of $\sigma^k(\mathsf{A})$ for any k.

Finally $v \in \mathcal{V}(\mathsf{A})$ is an **epivertex** if and only if: v is not incident to any $e \in \mathcal{E}(\mathsf{A}^-)$ in $\sigma(\mathsf{A}^-)$.

Note that if $v \in \mathcal{V}(\mathsf{A})$ is in the interior of $\sigma(\mathsf{A}^-)$, it is incident to some $e \in \mathcal{E}(\mathsf{A}^-)$ in $\sigma(\mathsf{A}^-)$. The endo-, meso- and epi- vertices of tiles, supertiles and tilings are the images of the endo-, meso- and epi- vertices of prototiles.

Terminals are another name for endo- and meso- vertices.

In figure 5 a substitution on the dimer tile ([7, 11]) is shown (the letters on the tiles are flipped around to indicate the isometries needed to assemble the supertiles). On the right of the figure the vertices are shown. Endo-, meso- and epi- vertices are marked. (Labels in $\mathcal{S}$ and $\mathcal{T}$ have also been assigned.)

Lemma 1.4 *If a vertex is a mesovertex for a given n-level supertile, the vertex is not coincident to a mesovertex for any descendant or ancestral supertile. Moreover, the vertex is coincident to an epivertex for every descendant supertile incident to the vertex.*

Proof Let v be a mesovertex for some prototile $\mathsf{A} \in \mathcal{S}$.

For every k, $0 \leq k \leq n$, $\sigma^n(v)$ is coincident to some vertex $\sigma^k(v_k) \in \sigma^k(A_k) \subset \sigma^n(\mathsf{A})$.

Since v is a mesovertex, no edge meets $\sigma^n(v)$ in the interior of $\sigma^n(\mathsf{A})$. Consequently, v_k is an epivertex of A_k.

Second, suppose v is coincident to some vertex $\sigma^n(v_n)$ of some ancestral supertile $\sigma^n(\mathsf{A}_n)$. Because v is a mesovertex, v is incident to some edge $e \in \sigma(A^-) \subset \sigma^n(A_n)$. Thus, $\sigma^n(\mathsf{A}_n)$ is a mesovertex or endovertex.

□

Lemma 1.5 *In a tiling in ($\mathcal{T}$, σ, $\mathcal{S}$), any vertex v of any prototile* A *in the interior of some supertile $\sigma^m(\mathsf{B})$ is either a mesovertex for some supertile contained in $\sigma^m(\mathsf{B})$ and containing* A*, or for some $n \in \mathbb{N}$, $\sigma^n(v) \subset \sigma^n(\mathsf{A})$ is coincident to an edge in the interior of $\sigma^n(\mathsf{A})$.*

Proof Let BA be a tile in the tiling such that BA is in the interior of some supertile. Let $\lambda_{\mathcal{A}}(\mathsf{BA}) = ...\mathsf{X}_n...\mathsf{X}_1\mathsf{A}$.

Let v be a vertex of prototile A such that for no $n \in \mathbb{N}$, $\sigma^n(v) \subset \sigma^n(\mathsf{A})$ is coincident to an edge in the interior of $\sigma^n(\mathsf{A})$. Thus v is either a mesovertex or an epivertex of A. A itself is a supertile, so in the first case we are done.

So suppose v is an epivertex of A. Now there exists some N, such that for all $n > N$, v is in the interior of $\sigma^n(\mathsf{X}_n)$ and for all $n \leq N$, v is on the boundary of $\sigma^n(\mathsf{X}_n)$. There exists some maximal $M \leq N$ such that for all $n \leq M$, v is on the boundary of $\sigma^n(\mathsf{X}_n)$ but not incident to any edge in $\sigma^n(\mathsf{X}_n)$. Thus v is a mesovertex or endovertex of $\sigma^M(\mathsf{X}_M)$. But for any n, $\sigma^n(v)$ is not coincident to an edge in the interior of $\sigma^{(n+M)}(\mathsf{X}_M)$; thus v is a mesovertex of $\sigma^n(\mathsf{X}_N)$.□

Note that the hypothesis holds for all vertices of tiles in a tiling with connected hierarchy.

2 Selecting structures in a substitution tiling

We now begin to select structures in our substitution tiling ($\mathcal{T}$, σ, $\mathcal{S}$) in order to define a labeling (Section 1.3).

There are a few main categories of label:

In Section 1 we described our initial categories $\mathcal{T}$, $\mathcal{S}$.

In Section 2.1 we define labels, in $\mathcal{V}$, $\mathcal{V}'$, $\mathcal{E}$, $\mathcal{E}'$, and $\mathcal{Z}$, concerning our first structure, **skeletons**. We also choose our important constant κ.

In Section 2.2, we construct **keys**, labels in $\mathcal{R}$, $\mathcal{V}''$ and $\mathcal{E}''$ that define the role of a supertile in the hierarchy.

In Section 2.3, we construct labels $\mathcal{W}$ concerning our second structure **wires**, and **wire keys** $\mathcal{U}$.

In Section 2.4, we construct **well-formed packets** in $\mathcal{Q}$ and $\mathcal{P}$, combinatorially complex combinations of previously defined labels; through our efforts here and in Section 3.3, we will have fairly simple matching rules (Section 3.5).

A simple summary of Section 2 is given in Section 2.5.

We begin with a given substitution system $\{\mathcal{T}, \sigma, \mathcal{S}\}$, and $\lambda_{\mathcal{A}}$ mapping each tile in each infinite-level supertile in each tiling to an (infinite-to-the right) address.

Crucially, note that even if a tiling admits more than one hierarchy, $\lambda_{\mathcal{A}}$ fixes a particular hierarchy, and every tile belongs to only one supertile of each level n in this fixed hierarchy.

2.1 Selecting structures to form skeletons

2.1.1 Selecting vertices $\mathcal{V}$ and edges $\mathcal{E}$

By assumption, there is a set of hereditary edges on the prototiles such that the tiling is sibling edge-to-edge. Let $\mathcal{E}_0(\mathsf{B})$ be such edges of $\mathsf{B} \in \mathcal{S}$. For each $\mathsf{A} \in \mathcal{T}$, let $\mathcal{E}(\mathsf{A})$ be the edges $\mathcal{E}_0(\mathsf{B})$, $\mathsf{B} \in \mathsf{A}^+$, that are not contained in the boundary of $\sigma(\mathsf{A})$. By sibling edge-to-edge we can take these edges as occurring in pairs, $+e, -e$. Let $\mathcal{E}$ be the disjoint union of the $\mathcal{E}(\mathsf{A})$, $\mathsf{A} \in \mathcal{T}$.

Given an $e \in \mathcal{E}$, we will take as implicit the element A in $\mathcal{T}$ such that $e \in \mathcal{E}(\mathsf{A})$, as well as the exact position of e in $\sigma(\mathsf{A})$. Note that in any configuration, the interiors of any $\sigma^n(e)$, $\sigma^m(e')$ are disjoint, $n \neq m$, $e \neq \pm e'$, $e, e' \in \mathcal{E}$.

For $e \in \mathcal{E}$ let $e+ = \mathsf{A} \in \mathcal{S}$ if $e \in \mathcal{E}_0(\mathsf{A})$

Let $\mathcal{V}(\mathsf{A})$, $\mathsf{A} \in \mathcal{S}$ be defined as the vertices of $\mathcal{E}_0(\mathsf{A})$. Note that these are hereditary and the tiling is sibling vertex-to-vertex by Lemma 1.3.

Let $\mathcal{V}$ be the disjoint union of the $\mathcal{V}(\mathsf{A})$ as A ranges over $\mathcal{S}$; in particular, an element v of $\mathcal{V}$ lies in a unique $\mathcal{V}(\mathsf{A})$.

Given a $v \in \mathcal{V}$, we will take as implicit the element A in $\mathcal{S}$ such that $v \in \mathcal{V}(\mathsf{A})$, as well as the exact position of v on the boundary of A. (This kind of implicitness will be usual as all these lists of labels are made).

In figure 5, $\mathcal{V}$ and $\mathcal{E}$ are shown for substitution on dimers [7, 11].

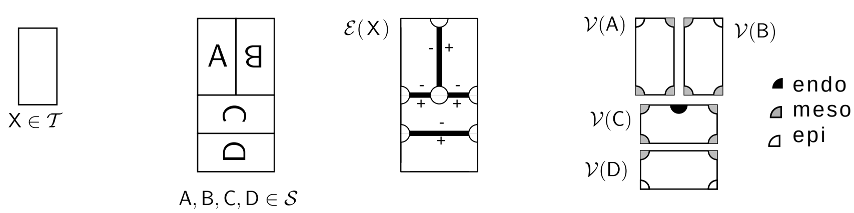

Figure 5: Edges $\mathcal{E}$ and vertices $\mathcal{V}$ of a dimer tiling

2.1.2 Selecting sites $\mathcal{Z}$

For each $\mathsf{A} \in \mathcal{S}$ we choose a number of **sites** that will link the skeleton of any supertile congruent to $\sigma^n(\mathsf{A})$ to the skeleton of the parent supertile congruent to $\sigma^{(n+1)}(\mathsf{A}^-)$ at edges congruent to $\sigma^n(\mathcal{E}(\mathsf{A}))$.

For each $e \in \mathcal{E}$, choose a natural number $\kappa(e)$ such that the supertile $\sigma^{\kappa(e)}(e+)$ contains an edge with an endpoint on $\sigma^{\kappa(e)}(e) \subset \sigma^{(\kappa(e)+1)}((e+)^-)$.

For each $\mathsf{A} \in \mathcal{S}$, we can choose a point $\mathcal{Z}_e$ on each edge $e \in \mathcal{E}(\mathsf{A})$ meeting $+e$ such that there is an edge in the interior of $\sigma^\kappa(+e)$ with one end at $\sigma^k(\mathcal{Z}_e) \subset \sigma^\kappa(e) \subset \sigma^{(\kappa+1)}(\mathsf{A})$. We take these points as **sites for the prototile** $+e \in \mathcal{S}$; in a supertile congruent to $\sigma^n(\mathsf{A})$, $\mathsf{A} \in \mathcal{S}$, each point corresponding to a $\sigma^n(\mathcal{Z}_e)$, $+e = \mathsf{A} \in \mathcal{S}$, is to be a **site for the supertile** .

In practice one might eliminate redundancies. First, sites as defined may lie at the endpoint of an edge. Thus two sites for a supertile may lie at the same point. We simply coalesce these into one site. Second, endovertices make excellent sites, since we will connect these to a supertile's skeleton anyway. These steps do nothing for the proof, but ease one's burden in practice. Let $\mathcal{Z}(\mathsf{A})$ be the collection of sites serving A.

We come to a technicality: $\mathcal{E}(\mathsf{A}^-)$ may be empty— this occurs if and only if A is a placeholder (Section 1.2). Sites need to be more carefully designed for such $\mathsf{A} \in \mathcal{S}$:

Because σ expands all distances, we must eventually subdivide the facets of our tiles; if A^- is a placeholder, there is an integer $j \geq 0$ and $\mathsf{B} \in \mathcal{S}$ such that: $\sigma^j(\mathsf{B})$ consists only of A but $\mathcal{E}(\mathsf{B}^-)$ is not empty. Then let $(\mathcal{Z}(\mathsf{A}))$ lie at the corresponding sites $\sigma^j(\mathcal{Z}(\mathsf{B})) \subset \mathsf{A}$.

2.1.3 Selecting κ

In a sense κ is the resolution at which we view the hierarchical structure.

For each $\mathsf{B} \in \mathcal{S}$ choose natural $\kappa(\mathsf{B})$ such that there is a connected collection of edges in $\sigma^{\kappa(\mathsf{B})}(\mathsf{B})$ that

i) contains every $\sigma^{(\kappa(\mathsf{B})-1)}(e)$ for $e \in \mathcal{E}(\mathsf{B})$,
ii) meets every $\mathcal{Z}(\mathsf{B})$ on the boundary of B and
iii) meets every endovertex on the boundary of B.

It is worth pointing out that we can indeed find such a $\kappa(\mathsf{B})$: $\kappa(\mathsf{B})$ is at least the maximum of the $\kappa(v)$ as v ranges over the endovertices of B and of the $\kappa(e)$, $e \in \mathcal{E}(\mathsf{B})$. To ensure that we can find a *connected* collection of edges, we may have to take further substitutions. Eventually however, such a $\kappa(\mathsf{B})$ can be found: σ expands all distances, all facets are bounded and fall into finite congruence classes; hence σ must eventually subdivide every k-facet, $0 < k \leq \mathsf{d}$.

Moreover, any $k > \kappa(\mathsf{B})$ will suffice as well, since edges are hereditary.

Take κ to be the maximum of these $\kappa(\mathsf{B})$.

In practice, κ is often rather low; in fact, many well known examples have $\kappa = 1$ and very few have $\kappa > 2$. For example, the Conway-Radin pinwheel [19], the Robinson triangles, the systems studied by Mozes [17], and numerous other examples allow $\kappa = 1$, as do most of the examples in [7]. In the current incarnation of our construction, the L-tiling requires $\kappa = 2$, leading to a set of markings that is far from optimal [12].

One can easily construct examples requiring arbitrarily large κ, such as Sadun's generalized pinwheel tilings [21]. Often a specific construction of matching rules can be finessed somewhat— in particular note that our bound κ is the maximum of many other bounds, each of which will play specific roles in the construction. (For example, placeholders do not, in practice, need to raise κ).

In figure 6, sites and κ have been chosen for the dimer tiling of figure 5. Note that we require $\kappa = 2$, both to find an appropriate site on one of the edges abutting prototile $\mathsf{A} \in \mathcal{S}$ (defined in figure 5), and so that we have a connected collection of edges that includes the image $\mathcal{E}(\mathsf{B})$ in each $\mathsf{B} \in \mathcal{S}$.

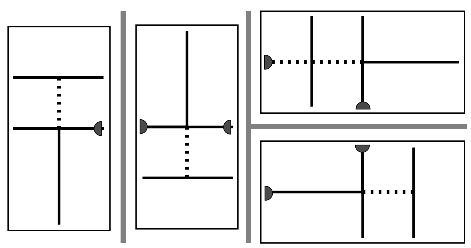

Figure 6: Sites, κ and skeletons for a dimer tiling

2.1.4 Selecting vertices $\mathcal{V}'$ and edges $\mathcal{E}'$ of skeletons

Lemma 2.1 *Given $(\mathcal{T}, \sigma, \mathcal{S})$, using hereditary edges $\mathcal{E}$, there exists $\kappa \in \mathbb{N}$, and sites $\mathcal{Z}$ such that*

for each $\mathsf{A} \in \mathcal{S}$ we may choose a **skeleton** *of j-level edges, $0 \leq j < \kappa$, $E_\mathsf{A} \subset \sigma^\kappa(\mathsf{A})$ such that*

i) $\cup_{e \in E(\mathsf{A})} \sigma^{(\kappa(e)-1)} \subset E_\mathsf{A}$;
ii) E_A *is connected;*
iii) $\sigma^\kappa(\mathcal{Z}(\mathsf{A})) \subset E_\mathsf{A}$;
iv) E_A *includes the endovertices of* $\sigma^\kappa(\mathsf{A})$.

Proof This immediately follows the definitions and existence of κ and $\mathcal{Z}$. $\square$

In order to describe the structure of this skeleton E_{A} we now subdivide our edges in E_{A} into edges $\mathcal{E}'(\mathsf{A})$ on the boundary of the images in $\sigma^{\kappa}(\mathsf{A})$ of the supertiles $(\sigma')^{\kappa}(\mathsf{B})$, $\mathsf{B} \in \mathsf{A}^{+}$ under the inflation σ^{-1}.

We require that: Each edge $\sigma^{j}(e)$, $0 \leq j < \kappa$ contained in E_{A} is tiled with edges in $\mathcal{E}'(\mathsf{A})$; and any facet of an edge in $\mathcal{E}'(\mathsf{A})$ is a facet of any edge in $\mathcal{E}'(\mathsf{A})$ to which it is incident.

We strongly require hereditary edges to ensure such an $\mathcal{E}'(\mathsf{A})$ exists: we could simply take appropriate $(\mathsf{d}-1)$-facets in $\sigma^{\kappa}(\mathsf{A})$; typically, this is choice far from optimal.

We define the set $\mathcal{V}'(\mathsf{A})$ of vertices of E_{A} to be the set of points coincident to vertices of the edges in $\mathcal{E}'$, union the points coincident to $\sigma^{(\kappa-1)}(z)$, $z \in \mathcal{Z}(\mathsf{B}) \subset E_{\mathsf{A}}, \mathsf{B} \in \mathsf{A}^{+}$, union the points coincident to $\sigma^{\kappa}(\mathcal{V}(\mathsf{A})$.

Suppose an edge $f \in \mathcal{E}'(\mathsf{A})$ was derived from some $\sigma^{j}(e)$, $e \in \mathcal{E}$; then the **level** of $\sigma^{n}(f)$ will be $(n+j)$.

Let $\mathcal{V}'$, $\mathcal{E}'$ be the disjoint union of the $\mathcal{V}'(\mathsf{A})$, $\mathcal{E}'(\mathsf{A})$ over $\mathcal{S}$.

Given a $v \in \mathcal{V}'$, it is to be implicit for which $\mathsf{A} \in \mathcal{S}$ that $v \in \mathcal{V}'(\mathsf{A})$ and the exact position of v in $\sigma^{\kappa}(\mathsf{A})$. Given an $e \in \mathcal{E}'$, it is to be implicit for which $\mathsf{A} \in \mathcal{S}$ that $e \in \mathcal{E}'(\mathsf{A})$ and the exact position of e in $\sigma^{\kappa}(\mathsf{A})$.

Note that for any vertex $v \in \mathcal{V}'(\mathsf{A})$, $v \subset \sigma^{\kappa}(\mathsf{A})$, if v is incident to some $\sigma^{j}(\mathsf{B})$, $B \in \mathcal{S}$, $0 \leq j < \kappa$, v does not need to be coincident to some vertex $\sigma^{j}(w)$ of $\sigma^{j}(\mathsf{B})$, $w \in \mathcal{V}(\mathsf{B})$.

In figure 7 the skeletons of three generations of supertiles in the dimer tiling of figure 5 are shown. Note skeletons frequently overlap, but only those of parent and child.

2.2 Keys

Roughly, keys encode position and role relative to a few levels of the hierarchy.

2.2.1 Supertile keys $\mathcal{R}$

Edges of tiles in tilings in $(\mathcal{T}, \sigma, \mathcal{S})$ might serve up to κ distinct skeletons.

We specify the ways this might occur. For each $\mathsf{A} \in \mathcal{S}$ let $\mathcal{R}(\mathsf{A})$ be the collection of all **supertile keys**, finite addresses $\mathsf{X}_{\kappa}...\mathsf{X}_{1}$ where $x_{j} \in \mathcal{S}$, $\mathsf{X}_{(j+1)} = \mathsf{X}_{j}^{-}$ for $j < \kappa$, and $\mathsf{X}_{1} = \mathsf{A}$. Note that such a address could have been thought of as having $\kappa + 1$ elements, for X_{κ} implicitly specifies the label of its parent in $\mathcal{T}$. Let $\mathcal{R}$ be the disjoint union of these $\mathcal{R}(\mathsf{A})$ over $\mathcal{S}$. (To facilitate a certain technical point below, we also include an extra *null* label in $\mathcal{R}$; this label contains no combinatorial information.) Now a given $\mathsf{R} = \mathsf{X}_{\kappa}...\mathsf{A} \in \mathcal{R}(\mathsf{A})$ exactly specifies all skeletons for all ancestral supertiles that pass through any supertile congruent to $\sigma^{n}(\mathsf{A})$ with address $...\mathsf{R}...\bullet$.

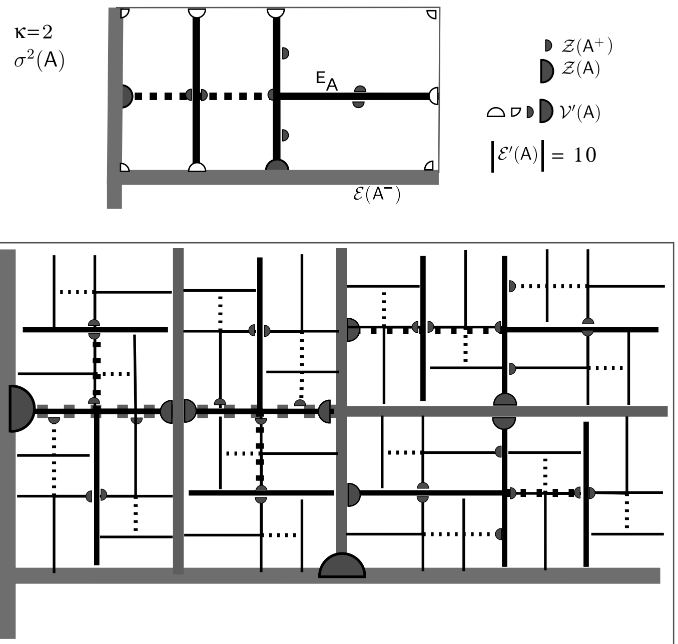

Figure 7: Skeletons of three generations of dimer

A label in $\mathcal{R}$ is the crucial information, then, that we will ensure that every supertile carries.

We begin defining maps λ_* to structures in a generic tiling with connected hierarchy. This will eventually be the basis for a labeling (Section 1.3) of the substitution tiling.

Lemma 2.2 *Let $\cup \mathsf{B}_i\mathsf{A}_i$ be a tiling in ($\mathcal{T}$, σ, $\mathcal{S}$), with $\lambda_{\mathcal{A}}$ (Section 1.2) chosen for $\cup \mathsf{B}_i\mathsf{A}_i$. Then there exists a map $\lambda_{\mathcal{R}} : \{\mathsf{B}\sigma^n(\mathsf{A}) \mid \mathsf{B} \in \mathcal{G}, n \in \mathbb{N}, \mathsf{A} \in \mathcal{S}, \mathsf{B}\sigma^n(\mathsf{A}) \subset \cup \mathsf{B}_i\mathsf{A}_i\} \to \mathcal{R}$ such that if $\lambda_{\mathcal{R}}(\sigma^n(\mathsf{A})) = \mathsf{X}_\kappa ... \mathsf{X}_1$ then $\mathsf{X}_1 = \mathsf{A}$ and for all $\mathsf{B} \in \mathsf{A}^+$, $\lambda_{\mathcal{R}}(\sigma^{(n-1)}(\mathsf{B}) \subset \sigma^n(\mathsf{A})) = \mathsf{X}_{(\kappa-1)} ... \mathsf{X}_1\mathsf{B}$, and if $\lambda_{\mathcal{A}}(\mathsf{B}_i\mathsf{A}_i) = ...\mathsf{X}_\kappa ... \mathsf{A}_i \bullet$, $\lambda_{\mathcal{R}}(\mathsf{B}_i\mathsf{A}_i) = \mathsf{X}_\kappa ... \mathsf{A}_i$.*

Proof In $\sigma^n(\mathsf{A}) \in \cup \mathsf{B}_i\mathsf{A}_i$, for any $\mathsf{B}_i\mathsf{A}_i, \mathsf{B}_j\mathsf{A}_j \subset \sigma^n(\mathsf{A})$, if $\lambda_{\mathcal{A}}(\mathsf{B}_i\mathsf{A}_i) = ...\mathsf{X}_k ... \mathsf{A}_i \bullet$ and $\lambda_{\mathcal{A}}(\mathsf{B}_j\mathsf{A}_j) = ...\mathsf{Y}_k ... \mathsf{A}_j \bullet$ then for all $k \geq n$, $\mathsf{X}_k = \mathsf{Y}_k$. Take $\lambda_{\mathcal{R}}(\sigma^n(\mathsf{A})) = \mathsf{X}_{(\kappa+n-1)} ... \mathsf{X}_n$. Clearly this meets the conditions of the lemma. $\square$

2.2.2 Edge keys $\mathcal{E}''$

We again redefine edges and vertices relative to the supertile keys. Take $\mathsf{R} = \mathsf{X}_\kappa ... \mathsf{X}_1 \in \mathcal{R}$. Recall that for $\mathsf{A} \in \mathcal{S}$, $\mathcal{E}'(\mathsf{A})$ and $\mathcal{V}'(\mathsf{A})$ are defined within $\sigma^\kappa(\mathsf{A})$.

We now subdivide the edges $\mathcal{E}(\mathsf{X}_1)$ to produce a new set of edges $\mathcal{E}''(\mathsf{R})$ for the prototiles $B \subset \sigma(\mathsf{X}_1)$, $\mathsf{B} \in \mathsf{X}_1^+$:

Let $\mathcal{E}_\kappa$ be the set of facets of the tiles in $\sigma^\kappa(\mathsf{X}_1)$ such that each facet in $\mathcal{E}_\kappa$ lies in some $\sigma^{(\kappa-1)}(e)$, $e \in \mathcal{E}(\mathsf{X}_1)$. Then, first: the edges $\mathcal{E}''(\mathsf{R})$ in $\sigma(\mathsf{X}_1)$ will be the union of facets of the form $\sigma^{-\kappa}(f)$, $f \in \mathcal{E}_\kappa$ (thus each edge $e \in \mathcal{E}(\mathsf{X}_1)$ is tiled with edges in $\mathcal{E}''(\mathsf{R})$).

Second we require of $\mathcal{E}''(\mathsf{R})$ that every facet of $\sigma^{(j-\kappa)}(e) \subset \sigma(\mathsf{X}_1) \subset \sigma^j(\mathsf{X}_j)$, $0 \le j \le \kappa$, $e \in \mathcal{E}'(\mathsf{X}_j)$, $\mathsf{X}_0 \in \mathsf{X}_1^+$ is the union of facets of edges in $\mathcal{E}''(\mathsf{R})$.

That is, we are careful to design structures at every facet at which the overlapping skeletons $\sigma^{(j-\kappa)}(E_{\mathsf{X}_j})$, $0 \le j \le \kappa$, $\mathsf{X}_0 \in \mathsf{X}_1^+$ meet the edges $\mathcal{E}(\mathsf{X}_1)$ in $\sigma(\mathsf{X}_1) \subset \sigma^\kappa(\mathsf{X}_\kappa)$.

Note such an $\mathcal{E}''(\mathsf{R})$ exists:

A poor choice is simply to take all $(\mathsf{d}-1)$-facets in $\sigma^{-k}((\sigma^\kappa(\mathsf{A})))$ (where the σ^{-k} is a inflation and the σ^κ is an iterated substitution).

Let $\mathcal{E}''$ be the disjoint union of the $\mathcal{E}''(\mathsf{R})$ over R.

Lemma 2.3 *Any edge f of any tile in the interior of some supertile in a tiling in ($\mathcal{T}$, σ, $\mathcal{S}$), is contained within a unique supertile edge $\sigma^n(e'')$, $e'' \in \mathcal{E}''$ or there exists $(\mathsf{R} = \mathsf{X}_n ... \mathsf{X}_1) \in \mathcal{R}$, $n < \kappa - 1$ such that f is a subset of a tiling by edges $\sigma^n(e_i)$, $e_i \in \mathcal{E}''(\mathsf{R})$. In this latter case, though, there is an $e \in \mathcal{E}(\mathsf{X}_1)$ such that $f, \sigma^n(e_i) \subset \sigma^n(e)$.*

Note that every edge of every tile in a tiling with connected hierarchy is in the interior of some supertile.

Proof Let f be an edge of a tile $\mathsf{B}_i\mathsf{A}_i$ in the interior of some supertile in the tiling. There is a minimal level supertile $\sigma^n(\mathsf{X}_1)$ containing this tile such that the interior of f is in the interior of $\sigma^n(\mathsf{X}_1)$. Thus f lies on the boundary of $\sigma^{(n-1)}(\mathsf{B})$ for some $\mathsf{B} \in \mathsf{X}_1^+$ and in $\sigma^n(e)$ for some $e \in \mathcal{E}(\mathsf{X}_1)$. The elements of $\mathcal{E}''$ are the unions of the images under $\sigma^{-\kappa}$ of the edges of tiles; thus f lies in the union of the images under $\sigma^{(n-1+\kappa)}$ of edges of tiles. In particular, since edges are hereditary, if $n \ge \kappa - 1$, f lies in the image of one edge of one tile and so lies in one $\sigma^n(e'')$; otherwise f is a subset of a tiling by edges $\sigma^n(e_i)$, $e_i \in \mathcal{E}''(\mathsf{R})$ for some $\mathsf{R} \in \mathcal{R}$. By definition, each $\sigma^n(e_i)$ lies in $\sigma^n(e)$. □

Lemma 2.4 *Any edge f of any tile not in the interior of any supertile in a tiling in ($\mathcal{T}$, σ, $\mathcal{S}$), is contained within a unique infinite fault-line*

Proof Let f be such an edge. f lies on the exterior of some tile, contained in a nested series $\{\sigma^n(\mathsf{A}_n)\}$ of supertiles; f must lie on the exterior of each $\sigma^n(\mathsf{A}_n)$ and so in the exterior of $\cup\sigma^n(\mathsf{A}_n)$ and hence on an infinite fault-line. □

Thus we define, for any tiling $\cup \mathsf{B}_i \mathsf{A}_i$ in $(\mathcal{T}, \sigma, \mathcal{S})$, with $\mathcal{E}''$ derived in Section 2.2.2, a map $\lambda_{\mathcal{E}''} : E \to \mathcal{E}'' \cup \{\mathsf{null}\}$ where E is the set of all points of all edges of tiles in the tiling: for each x a point in some edge f of some tile in the tiling, if f is in the interior of some supertile, let $\lambda_{\mathcal{E}''}(x)$ be the $e'' \in \mathcal{E}''$ or $e_i \in \mathcal{E}''$ produced by Lemma 2.3 such that x lies in $\sigma^n(e'')$ or $\sigma^n(e_1)$; otherwise let $\lambda_{\mathcal{E}''}(f) = \mathsf{null}$.

2.2.3 Vertex hulls $\mathcal{V}''$

Take $\mathsf{R} = \mathsf{X}_\kappa ... \mathsf{X}_1 \in \mathcal{R}$, and let $\mathcal{V}''(\mathsf{R})$ be the images of the intersection with $\sigma(\mathsf{X}_1)$ of exceedingly small δ balls centered at each vertex of $\sigma(\mathsf{X}_1)$ and at the points coincident to endpoints of the elements of $\mathcal{E}''(\mathsf{R})$. The elements of $\mathcal{V}''$ are called **vertex hulls**; the

center of a hull in $\mathcal{V}''(\mathsf{R})$ is the point at which the δ ball defining the hull is centered.

Lemma 2.5 *The set of centers of the $\mathcal{V}''(\mathsf{R})$ is the set of all points v in $\sigma(\mathsf{X}_1)$ such that:*

v is an endpoint of of some $\sigma^{(j-\kappa)}(e)$, $e \in \mathcal{E}'(\mathsf{X}_j)$, $1 \le j \le \kappa$ with $\sigma^{(j-\kappa)}(v) \in f$ for some $f \in \mathcal{E}(\mathsf{X}_1)$,

or $v = \sigma(w)$, $w \in \mathcal{V}(\mathsf{X}_1)$,

or $v = \sigma^{-\kappa}(z)$, $z \in \mathcal{Z}(\mathsf{A})$, $\mathsf{A} \in \mathsf{X}_1^+$.

Proof This follows the definitions. □

.

That is, for $\mathcal{V}''$ we take the endpoints of any edges in higher level skeletons meeting the 0-level edges in X_1, the images of the vertices of X_1 under σ^1, and the images of the sites of the daughter tiles on the edges of X_1.

Place an arbitrary ordering in $\mathcal{V}''$. Now we can assume that the 1-facets of each edge in $\mathcal{E}''$ are connected. Define for each $e \in \mathcal{E}''$ the set $\mathcal{F}(e)$ of labels of 1-facets. Each endpoint of an element of $\mathcal{F}(e)$ is associated with a label in $\mathcal{V}''$; we associate each label in $\mathcal{F}(e)$ with an arrow pointing from the lower endpoint to the higher. Let $\mathcal{F}$ be the disjoint union over $\mathcal{E}''$ of the $\mathcal{F}(e)$.

We mark each hull in $\mathcal{V}''$ with information concerning the edges it meets:

First the hull is **darkly marked** with the positions, orientations, labels in $\mathcal{E}'(\mathsf{X}_j)$ and the relative level j of the various edges in $\sigma^{(j-\kappa)}(e)$, $e \in \mathcal{E}'(\mathsf{X}_j)$, $1 \le j \le \kappa$ incident to the vertex in $\sigma(\mathsf{X}_1)$, as well as any edges $\sigma^{(-\kappa)}(e)$, $e \in \mathcal{E}'(A)$, $A \in \mathsf{X}_1^+$ for which v incident to v. Note that a darkly marked edge in a vertex hull may end up with many edge markings in $\mathcal{E}'$; however, we take the single edge of lowest relative level as our marking, for this will be sufficient.

Second, given $v \in \mathcal{V}''$, there is some finite N such that if $\sigma^N(v)$ is incident to an edge e in $\sigma^{(N+1)}(\mathsf{X}_1)$, then for any $n > N$, $\sigma^n(v)$ is incident to $\sigma^{(n-V+1)}(e)$ in $\sigma^{(n+1)}(\mathsf{X}_1)$, because of the finite valence of the vertices in our substitution tiling. That is, we can **lightly mark** the positions, orientations, and labels in E' of all lower level edges that are eventually incident to v. We also put a *height* function on these lightly marked edges: suppose e, f are lightly marked edges incident to v such that there is some n such that $\sigma^n(v)$ is incident to an image of e but not incident to an image of f; then e is **higher** than f.

On the 1-facets of all the edges, lightly marked or darkly marked, we further mark the vertex hull with appropriate labels and orientations in $\mathcal{F}$. (When $\mathsf{d} = 2$ we simply place arrows on the elements of $\mathcal{E}''$).

Note that this hull will be either an d-ball or a sector of a d-ball. Because we have assumed sibling vertex-to-vertex, the first case always occurs if v is in the interior of $\sigma(\mathsf{A})$; the latter case will occur only on the boundary of $\sigma(\mathsf{A})$. In the latter case define the **flat sides** of the hull to be image of the boundary of the $\sigma(\mathsf{A})$ on the boundary of the hull (i.e. the planes on which the d-ball was cut to make a sector). (Here sibling vertex-to-vertex is not really being used very strongly; to drop the condition we merely have to delineate a second kind of flat side).

Given $v \in \mathcal{V}''(\mathsf{X}_\kappa...\mathsf{X}_1)$, its hull will be implicit. The hulls will orient a vertex in the tiling, showing in which directions edges are expected.

In figure 8, the construction of $\mathcal{E}''(AC)$ and $\mathcal{V}''(AC)$ is indicated for the dimer tiling of figure 5 and figure 5. It is important to note that these hulls actually have much more information encoded within them– in particular *specific* labels associated with these particular spots in the structure. Also light and dark edges are not distinguished in this figure.

Note that $\mathcal{E}''(\mathsf{X}_\kappa...\mathsf{X}_1)$, $\mathcal{V}''(\mathsf{X}_\kappa...\mathsf{X}_1)$ are defined as structures in $\sigma(\mathsf{X}_1)$. Also note that any edge, of level greater than κ, of a tile in a tiling has a unique label in $\mathcal{E}''$ (whereas it may have up to κ labels in $\mathcal{E}'$. Also, if the edge is of level less than κ, the construction of $\mathcal{E}''$ may have divided it into some finite number of pieces). We can also mark terminals as appropriate on the vertex hulls, if they are at the end of a dark edge of relative level-1, on the exterior of $\sigma(\mathsf{X}_1)$.

Labels $v \in \mathcal{V}''(\mathsf{R})$ $e \in \mathcal{E}''(\mathsf{R})$, with $\mathsf{R} = \mathsf{X}_\kappa...\mathsf{X}_1 \in \mathcal{R}$, exactly specify what role, if any, $\sigma^n(v)$ and $\sigma^n(e)$ play in skeletons serving supertiles $\sigma^{(n-1+j)}(\mathsf{X}_j)$, $1 \leq j \leq \kappa$; that is, a edge or vertex label $e \in \mathcal{E}''(\mathsf{X}_\kappa...\mathsf{X}_1)$, $v \in \mathcal{V}''(\mathsf{X}_\kappa...\mathsf{X}_1)$ exactly specifies whether the edge or vertex has a label in each $\mathcal{E}'(\mathsf{X}_j)$, $\mathcal{V}'(\mathsf{X}_j)$ and if so what this label is. Note however that because vertices may lie on the boundary of the supertile they serve, a given vertex may lie in several unrelated skeletons. This is discussed further in the construction of vertex decorations.

We will return to these classes momentarily. The main point here is that the primary information associated with each supertile is to be its label in $\mathcal{R}$. With this specified, the role of the supertile in all higher level skeletons is fixed.

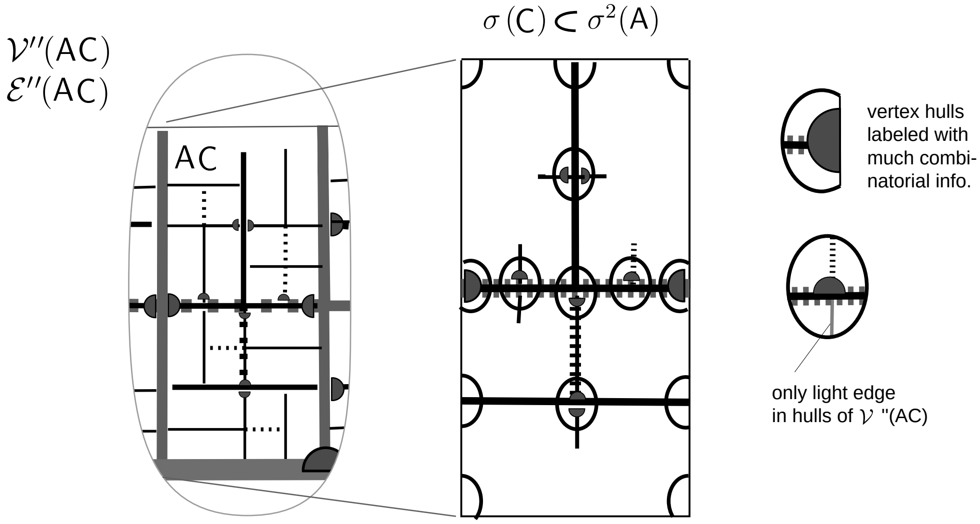

Figure 8: elements of $\mathcal{V}''$, $\mathcal{E}''$ and vertex hulls

2.3 Vertex-wires $\mathcal{W}$ and wire-keys $\mathcal{U}$

In the proof of the matching rules, we will need each supertile's key to arrive at three kinds of points on the boundary of the supertile: sites, endovertices, and mesovertices. The first two kinds of points lie on the supertile's skeleton, and thus are dealt with.

We require a second, independent structure—**vertex wires**— to send information to the mesovertices of a supertile; more generally we can deliver information to any **rational address**:

A point in a tile A has **rational address** if and only if there exist strings $\mathsf{X} = \mathsf{X}_{n+m} \dots \mathsf{X}_{1+m}$, $\mathsf{Y} = \mathsf{Y}_m \dots \mathsf{Y}_1$ of digits in S such that in $\sigma^{k+1}(\mathsf{A})$, the address of the tile containing x is $\mathsf{AA}_k \dots A_1\bullet$ where for $j > k-n-m$, $A_j = X_{n+m-k+j}$, and for $j \leq k-n-m$, $A_j = X_{n+m-k+j(mod(m))}$.
That is, if we allow ourselves the addressing of points as described in [6], a point x has rational address if and only if it has an address of the form $\mathsf{A}\bullet\mathsf{XYYY}\dots$

Recall that a mesovertex $v \in \mathcal{V}(\mathsf{A})$, $\mathsf{A} \in \mathcal{S}$ lies on boundary of A and at the end of an edge in $\mathcal{E}(\mathsf{A}^-)$, but that for no n does $\sigma^n(v)$ lie at the end of any edge in the interior of $\sigma^n(\mathsf{A})$. For each $\mathsf{A} \in \mathcal{S}$, let $\mathcal{V}_m(\mathsf{A}) \subset \mathcal{V}(\mathsf{A})$ be the collection of mesovertices of A, and let $\mathcal{V}_m$ be the disjoint union of the $\mathcal{V}_m(\mathsf{A})$ as A ranges over S.

Lemma 2.6 *Let v be a vertex of a prototile* A*. Then v has rational address.*

Proof For all $n \in \mathbb{N}$, $\sigma^n(v)$ is incident to exactly one tile, congruent to $\mathsf{X}_n \in \mathcal{S}$, and is coincident with a vertex $v_n \in \mathcal{V}(\mathsf{X}_n)$ of this tile. Note that the address of v is thus $\mathsf{A}\bullet\mathsf{X}_1\mathsf{X}_2$.... Since $\mathcal{V}$, $\mathcal{S}$, are finite and vertices are hereditary, this address must be rational, i.e., of the form $\mathsf{A}\bullet\mathsf{XYYY}$... where X, Y are finite addresses. (In fact, this is precisely the single point in the construction at which hereditary vertices are invoked) □

Let $v \in \mathcal{V}_m(\mathsf{A})$. There are non-negative integers $l(v)$, $m(v)$ such that we may suppose $\mathsf{X} = \mathsf{X}_1...\mathsf{X}_{l(v)}$ and $\mathsf{Y} = \mathsf{X}_{(l(v)+1)}...\mathsf{X}_{m(v)}$, and that for $j, k > l(v)$, if $k = j\ (mod(m(v) - l(v)))$, then $\sigma^k(x_j) = \sigma^j(x_k)$ (taking v as the fixed origin), and $(v_j) = (v_k) \in \mathcal{V}$. That is, we require that v_k occupies the same position and orientation in X_k as does v_j in X_j.

Let $\mathcal{W}(v)$ be this sequence $[(\mathsf{A} = \mathsf{X}_0)\bullet\mathsf{X}_1, ..., \mathsf{X}_{l(v)}, \mathsf{X}_{(l(v)+1)}, ...\mathsf{X}_{m(v)}]$. We will take as implicit, given $\mathsf{X}_i \in \mathsf{W}(v)$, the values of $l(v)$ and $m(v)$, the label $v_i \in \mathcal{V}(\mathsf{X}_i)$, the unique successor $\mathsf{X}_{(i+1)}$ (or, if $i = m(v)$, $\mathsf{X}_{(l+1)}$) and (for $i \neq (l+1)$) the unique predecessor $\mathsf{X}_{(i-1)}$, and for $\mathsf{X}_{(l+1)}$ the two predecessors x_l and x_m. To make indexing easier, for any $n > m$, X_n will be taken to mean X_k where $l < k \leq m$ and $n = k\ (mod(m-l))$. Define for each X_i a set of **possible preceders**– strings drawn from XYYY... ending in X_i, either with length κ or with length less than κ and beginning with X_1.

$\mathcal{W}(v)$ is the **vertex-wire** for the mesovertex v. Let $\mathcal{W}$ be the disjoint union over all mesovertices of the $\mathcal{W}(v)$. As usual, given a $w \in \mathcal{W}$, it is implicit for which v that w is an element of $\mathcal{W}(v)$.

We next define for any tiling in $(T, \sigma, \mathcal{S})$ a map $\lambda_{\mathcal{U}} : V \to (\mathcal{W} \times \mathcal{R}) \cup \{\mathsf{null}\}$ where V is the set of all epivertices of the supertiles in the tiling:

Recall Lemma 1.5.

In particular, any epivertex v of a supertile $\sigma^j(\mathsf{B})$ in a tiling is either on the exterior of any supertile to which it is incident or either incident some lower level edge in the interior of the supertile, or is coincident to a mesovertex w of some higher level supertile $\sigma^n(\mathsf{A})$, with $\lambda_{\mathcal{R}}(\sigma^n(\mathsf{A})) = \mathsf{R} \in \mathcal{R}$. In the first and second cases, take $\lambda_{\mathcal{U}}(v) = \mathsf{null}$. Otherwise take $\lambda_{\mathcal{U}}(v) = (\mathsf{X}_{(n-j)}, \mathsf{R})$ where $\mathsf{X}_{(n-j)}$ is the $(n-j)$th digit of the vertex wire $\mathcal{W}(w)$.

The possible images of epivertices are **wire-keys**; let $\mathcal{U}(v) \subset (\mathcal{W} \times \mathcal{R})$, be the set of possible wire keys of a particular epivertex $v \in \mathsf{A} \in \mathcal{S}$, and let $\mathcal{U}$ be the disjoint union of the $\mathcal{U}(v)$.

2.4 Supertile packets $\mathcal{Q}$, edge- and vertex-packets $\mathcal{P}$

We now summarize the classes of labels associated with each primary structure. *Packets* will be certain bundles of labels associated with various structures. **Edge packets** and **vertex packets** will, in essence, be the markings for our new tiles.

A given supertile $\sigma^n(\mathsf{A})$, $\mathsf{A} \in \mathcal{S}$ will carry a key $\mathsf{R} \in \mathcal{R}(\mathsf{A})$ of its own and wire keys $\mathsf{U} \in \mathcal{U}(v)$ for each of epivertex v in $\mathcal{V}(\mathsf{A})$. For each $\mathsf{A} \in \mathcal{S}$, let $\mathcal{Q}(\mathsf{A})$ be the **supertile packets**– sets of labels for supertiles $\sigma^n(\mathsf{A})$: that is sets of the form $[\mathsf{R}, \mathsf{U}_1, \mathsf{U}_2, ...]$ such that $\mathsf{R} \in \mathcal{R}(\mathsf{A})$ and there is exactly one $\mathsf{U}_i \in \mathcal{U}(v)$ for each epivertex v in $\mathcal{V}(\mathsf{A})$ and each U_i lies in some $\mathcal{U}(v)$ for some epivertex v in $\mathcal{V}(\mathsf{A})$. Note these sets of labels are finite and there are finitely many such sets.

Take $\mathcal{Q}$ to be the disjoint union of the $\mathcal{Q}(\mathsf{A})$ over $\mathcal{S}$. We also include an extra **null** label in $\mathcal{Q}$.

Given a tiling $\cup \mathsf{B}_i\mathsf{A}_i$ in $(T,\ \sigma,\ \mathcal{S})$, define a map $\lambda_{\mathcal{Q}} : \{\sigma^n(\mathsf{A}) \mid n \in \mathbb{N}, \mathsf{A} \in \mathcal{S}, \sigma^n(\mathsf{A}) \subset \cup \mathsf{B}_i\mathsf{A}_i\} \to \mathcal{Q}$ with, for supertile $\sigma^n(\mathsf{A})$ with epivertices v_i, $\lambda_{\mathcal{Q}}(\sigma^n(\mathsf{A})) = [\lambda_{\mathcal{R}}, \lambda_{\mathcal{U}}(v_1 \in \sigma^n(\mathsf{A})), ...]$.

Lemma 2.7 *Let* $[\mathsf{R}, \mathsf{U}_1, \mathsf{U}_2, ...]$ *be the image under* $\lambda_{\mathcal{Q}}$ *of some supertile in a tiling with connected hierarchy. For each non-null* $\mathsf{U}_i = [\mathsf{Y}_{n_i}, \mathsf{R}_i]$ *with* $\mathsf{Y}_{n_i} \in \mathsf{W}(v)$, *note* R *is either a possible preceder of* Y_{n_i} *or terminates in a possible preceder of length* $k < \kappa$ *of* Y_{n_i}. *In the latter case, we further note the last* $(\kappa - k)$ *digits of* R_i *are the first* $(\kappa - k)$ *digits of* R.

Proof This follows immediately from the definitions. □

The defining label for each edge of every tile is a label in $\mathcal{E}''$. Thus a given edge of a tile will convey a **edge packet** $[e, \mathsf{Q}_1, \mathsf{Q}_2, \ldots\ \mathsf{Q}_\kappa] \in (\mathcal{E}'' \times \mathcal{Q}^\kappa) \cup \{\mathsf{null}\}$ of labels, consisting of its own label $e \in \mathcal{E}''(\mathsf{X}_\kappa ... \mathsf{X}_1) \in \mathcal{E}''$, and a finite collection of labels in $\mathcal{Q}$. The first element, e, will be the **header** of the packet; the rest will be the **trailer**.

Given a tiling in $(T,\ \sigma,\ \mathcal{S})$, let E be the set of all points of edges of tiles in the tiling and define $\lambda_{\mathcal{P}} : \mathcal{E} \to (\mathcal{E}'' \times \mathcal{Q}^\kappa) \cup \{\mathsf{null}\}$:

Recall each point $f \in E$ of each edge either lies on an infinite fault line or lies in some $\sigma^n(e)$, $e \in \mathcal{E}''(\mathsf{X}_\kappa ... \mathsf{X}_1)$, and thus lies within up to κ skeletons $\sigma^{(n+k-1)}(E_{\mathsf{X}_k})$, $\kappa \in K \subset \{1, ..., \kappa\}$, with $1 \in K$ (This is given by $\lambda_{\mathcal{E}''}(f)$). In the first case let $\lambda_{\mathcal{P}}(f) =$null; in the second let $\lambda_{\mathcal{P}}(f) = [e, \mathsf{Q}_1, \mathsf{Q}_2, ..., \mathsf{Q}_\kappa]$, where $\mathsf{Q}_i = \lambda_{\mathcal{Q}}(\sigma^{(n+k-1)}(\mathsf{X}_k))$ for $i \in K$ and $\mathsf{Q}_i =$null otherwise.

The image of E under $\lambda_{\mathcal{P}}$ will be **edge packets** $\mathcal{P}$.

Thus given edge will carry a packet in $\mathcal{P}$, which amounts to a label in $\mathcal{E}''$ for the lowest level skeleton to which the edge belongs and κ keys for supertiles of the next κ levels in the hierarchy, and any wire keys these supertiles are in turn transmitting.

The following amounts to a long lemma with trivial proof following immediately from the construction:

Any packet in $[e, \mathsf{Q}_1,\ \mathsf{Q}_2,\ ...\ \mathsf{Q}_\kappa] \in \mathcal{P}$, $e \in \mathcal{E}''(\mathsf{X}_\kappa...\mathsf{X}_1)$ satisfies:

Let each non-null $\mathsf{Q}_i = [\mathsf{R}_i, \mathsf{U}_{i1}, \mathsf{U}_{i2}, ...]$. We must have $R_1 = \mathsf{X}_\kappa...\mathsf{X}_1$.

For each pair of non-null Q_i, Q_j, $\mathsf{R}_j = \mathsf{X}_{(\kappa,j)}...\mathsf{X}_{(1,j)}$, $\mathsf{R}_i = \mathsf{X}_{(\kappa,i)}...\mathsf{X}_{(1,i)}$ with $1 \leq j < i \leq \kappa$ in a packet for label $e \in \mathcal{E}''$, we have $\mathsf{X}_{(k-i,i)} = \mathsf{X}_{(k-j,j)}$ for $i < k \leq j + \kappa$ and $e \in \mathcal{E}''(R_1)$. That is, we can define, for well formed edge packets, a **longer address** $\mathsf{X}_j...\mathsf{X}_\kappa...\mathsf{X}_1$ of some length j, $\kappa \leq j \leq (2\kappa - 1)$ such that for any $1 \leq k \leq j$, $1 \leq i \leq k$, $\mathsf{X}_k = \mathsf{X}_{(i,k-i)}$; the R_i are in effect snippets of this longer address. (The length j depends on which R_j and Q_j are not vacant, that is, on which higher level skeletons e serves.)

Given the longer address $\mathsf{X}_j...\mathsf{X}_\kappa...\mathsf{X}_1$, it is clear by examining $\sigma^j(\mathsf{X}_j)$ exactly how the vertices of any X_k, $1 \leq k \leq \kappa$ coincide with vertices of any other X_l, $1 \leq l \leq \kappa$. We note the U_{ij} do not conflict at these vertices:

In particular, suppose v_k is an epivertex of X_k, with corresponding U_{ki}, incident to an epivertex v_l of X_l, with corresponding U_{li}. Then U_{ki} is null in $\mathcal{U}(v_k)$ if and only if U_{li} is null in $\mathcal{U}(v_l)$; if neither is null, let $\mathsf{U}_{ki} = (\mathsf{Y}_{n_{ki}} \in \mathsf{W}(v_{ki}), \mathsf{R}_{ki})$, and $\mathsf{U}_{li} = (\mathsf{Y}_{n_{li}} \in \mathsf{W}(v_{li}), \mathsf{R}_{li})$. Then we must have $v_{ki} = v_{li} \in \mathcal{V}_m$, $n_{ki} + l = n_{li} + k\ (mod(m(v_{ki}) - l(v_{ki})$ and $\mathsf{R}_{ki} = \mathsf{R}_{li}$.

Suppose v_k is an epivertex of X_k, with corresponding U_{ki}, incident to some mesovertex v of v_l of X_l; then U_{ki} must not be null. Let $\mathsf{U}_{ki} = (\mathsf{Y}_{n_{ki}} \in \mathsf{W}(v_{ki}), \mathsf{R}_{ki})$. We require $\mathsf{R}_l = \mathsf{R}_{ki}$, $v = v_{ki}$, and $\mathsf{Y}_{n_{ki}}$, the (n_{ki})th digit of $W(v_{ki})$.

We are not particularly concerned with endovertices incident to v_k at this moment.

Note that for supertiles $\sigma^{n-1}(\mathsf{B}) \subset \sigma^n(\mathsf{A})$, $\mathsf{B} \in \mathsf{A}^+$, the above conditions on the Q_i satisfied by a well-formed edge packet will hold, taking $\mathsf{Q}_1 = \lambda_{\mathcal{Q}}(\sigma^{n-1}(\mathsf{B}))$ and $\mathsf{Q}_2 = \lambda_{\mathcal{Q}}(\sigma^n(\mathsf{A}))$.

If $\kappa = 1$, we can compare the packet $\mathsf{Q}_1 = [\mathsf{A}, \mathsf{U}_{11}, \mathsf{U}_{12}, ...]$ conveyed by a daughter tile to the packet $\mathsf{Q}_2[\mathsf{B}, \mathsf{U}_{21}, \mathsf{U}_{22}, ...]$ of its parent. We will say Q_1 and Q_2 are **paired** if $\mathsf{A} \in \mathsf{B}^+$ and the conditions on the U_{ij} must be satisfied as above.

In practice, the well-formed packets are not so difficult to list, merely tedious. The restrictions simply ensure the packets are not so malformed as to never arise in an actual tiling. Let $\mathcal{P}(\mathcal{E}'')$ be the collection of these well-formed edge packets.

We similarly define **vertex packets** $[v, \mathsf{Q}_1,\ \mathsf{Q}_2,\ ...\ \mathsf{Q}_\kappa]$ of labels, consisting of a vertex label $v \in \mathcal{V}''(\mathsf{X}_\kappa...\mathsf{X}_1) \in \mathcal{V}''$, and a finite collection of labels in $\mathcal{Q}$. We only define such packets for vertices $\mathcal{V}''(\mathsf{X}_\kappa...\mathsf{X}_1)$ incident to the skeleton of X_1.

As above, let v be the **header** of the packet $[v, \mathsf{Q}_1,\ \mathsf{Q}_2,\ ...\ \mathsf{Q}_\kappa]$ and the Q_i the **trailer** of the packet. Note that our description of well-formed edge-packet made no real use that the first item in such a packet is a edge label in $\mathcal{E}''$; we thus make the same restrictions in defining well-formed vertex packets and let $\mathcal{P}(\mathcal{V}'')$ be the collection of all well formed vertex packets. For each element of $\mathcal{P}(\mathcal{V}'')$ we can also define a **longer address** as we did for the $\mathcal{P}(\mathcal{E}'')$.

A given edge will carry a packet in $\mathcal{P}$, which amounts to a label in $\mathcal{E}''$ for the lowest level skeleton to which the edge belongs and κ keys for supertiles of the next κ levels in the hierarchy, and any wire keys these supertiles are in turn transmitting.

Finally, we define, given a tiling in $(T, \sigma, \mathcal{S})$, a map $\lambda_{\mathcal{F}}$ mapping the points of each 1-facet of each edge of each tile of a tiling into $([0,1] \times \mathcal{F}) \cup \{\mathsf{null}\}$, as follows:

Every edge of every tile f in the tiling either lies on a infinite fault-line or lies in a unique $\sigma^n(e)$, $e \in \mathcal{E}''$. In the first case, take $\lambda_{\mathcal{F}}(h) = \mathsf{null}$ for each 1-facet of f.

In the second case every 1-facet h of f may or may not lie within a 1-facet of $\sigma^n(e)$. If it does not, take $\lambda_{\mathcal{F}}(h) = \mathsf{null}$. Otherwise, h lies within a 1-facet $\sigma^n(h')$ of $\sigma^n(e)$, $h' \in \mathcal{F}(e)$ (Section 2.2.3). Recall h' has an assigned orientation in e (an arrow pointing from one end of h to the other); let h inherit this orientation from $\sigma(e)$.

Thus h has a "high end" and a "low end", lying towards the head and tail, respectively, of the arrow on $\sigma(h')$. Let $\lambda_{\mathcal{F}}$ linearly map h to $[0,1] \times \{h'\}$ with the high end of h mapped to 1, the low to 0.

This last map will serve to orient our labeling in the tiling; recall that the set of 1-facets of the tiles in the tiling is connected.

2.5 Summary of selected structures

Thus, given $\{\mathcal{T}, \sigma, \mathcal{S}\}$, and $\lambda_{\mathcal{A}}$ mapping each tile in each tiling to an infinite-to-the right address, we provide the following well-defined maps:

i) First and foremost, supertiles are identified via $\lambda_{\mathcal{R}}$ with a supertile key in $\mathcal{R}$ (Lemma 2.2);

ii) Second, each epivertex of a supertile may be a mesovertex for at most one (Lemma 1.4) higher-level supertile. Thus we have $\lambda_{\mathcal{U}}$ (Section 2.3) relating each epivertex in the tiling to a wire-key in $\mathcal{U}$ (or null), relating the orientation of the epivertex to the orientation of the ancestral mesovertex and its supertile.

iii) Thus, each supertile is associated via $\lambda_{\mathcal{Q}}$ (Section 2.4) with a (well-formed) supertile packet in $\mathcal{Q}$— the supertile's key and the keys of its epivertices. This packet is to reside on the supertile's skeleton.

iv) Each edge in a tiling will lie on up to κ skeletons (serving κ consecutive levels of supertile) . Thus each edge is associated, via $\lambda_{\mathcal{P}}$ (Section 2.4) with an edge packet in $\mathcal{P}$, consisting of up to κ supertile keys and an edge key in $\mathcal{E}''$ that gives the edge's precise position and orientation in the skeletons.

v) Similarly, each vertex hull $\mathcal{V}''$ is marked with the edge packets of the edges it meets.

vi) Finally, orientations of objects are shuttled around the 1-facets of the tiling via $\lambda_{\mathcal{F}}$ (Section 2.4).

3 Creating tiles and markings

We will define new prototiles $\mathcal{T}'$ and matching rules $\mathcal{M}$ for piecing them together.

We will then explicitly define a labeling of ($\mathcal{T}$, σ, $\mathcal{S}$). In effect, on either side of each edge in a tiling we will mark with the appropriate edge packet; the middle of each tile will be marked with an appropriate supertile packet.

We will then show this new matching rule tiling ($\mathcal{T}'$, $\mathcal{M}$) reconstructs the labeling.

3.1 Creating new unmarked tiles

We will describe three flavors of proto-tile, and then markings derived from the various sets of labels. Here we use the finiteness of $\mathcal{V}$ to ensure that we can find a finite collection of standard tiles employing a finite collection of matching rules.

As a point of interest, we could merely take for $\mathcal{T}'$ the labelings of our original prototiles, but this technique is as well defined and results in a huge reduction in the number of tiles required.

In essence, we relegate largely independent pieces of information to separate tiles.

Lemma 3.1 *Given a substitution tiling (T, σ, $\mathcal{S}$), there exist $\epsilon, \delta > 0$ such that every point in any prototile of $\mathcal{T}$ that is less than ϵ from more than one 1-facet of a prototile is less than δ from a vertex of the prototile; and such that no point in a tile is within 2δ of more than one vertex of the tile.*

Note that $\mathsf{d} = 2$, the 1-facets of the prototile are the prototile's edges. When $\mathsf{d} > 2$, note that the lemma implies that if a point is within ϵ of more than one edge of the prototile, the point is within ϵ of a 1-facet of the prototile (recall the the $k-$facets of the prototile all lie within k-planes in $\mathbb{E}^{\mathsf{d}}$.

Proof This basically follows from the finiteness of $\mathcal{V}$ and $\mathcal{E}$ and that the elements of $\mathcal{E}$ are confined to planes in $\mathbb{E}^{\mathsf{d}}$. □

We define a set $\lambda^{-1}(\partial)$ of points in each tiling as follows (this set will both define the boundaries of our new tiles $\mathcal{T}'$ and begin to define our labeling λ):

First, any point in any tile in any supertile in any tiling that lies on the boundary of the tile, exactly δ from a vertex of a tile or exactly ϵ from one edge or equidistant to two or more edges of the tile, such that this distance is less than ϵ and no edge is closer than δ to any vertex of the tile will be labeled "∂". This will mark the boundaries of our new tiles.

Note that the closures of the components of any tiling less the points marked ∂ fall into a finite collection of congruence classes: we partition these classes into **unmarked vertex hulls**, **edge tiles**, and **small tiles**, depending on if they were originally within δ of a vertex, ϵ of an edge but further than δ from a vertex, or further than ϵ from an edge and δ from a vertex.

Note of course that the unmarked vertex hulls are exactly that– congruent to vertex hulls in $\mathcal{V}''$ with the markings wiped off.

We will define **vertex-tiles** as the union of such hulls shortly; vertex-tiles will be marked d-balls (**disks**) or sectors of d-balls.

Because of problems with vertices and edges of level less than zero, we will ultimately coalesce small tiles into **big tiles**, derived from supertiles of level κ.

In figure 9 the triangle tiling of figure 3 has been carved up into small tiles, edge tiles and vertex hulls. The vertex hulls have not yet been assembled into vertex tiles. None of these tiles have yet been marked.

The **out-side** of an edge-tile is the image of the boundary of the original tile from whence it was derived; the **in-side** of an edge-tile are the points on the boundary of the edge-tile that are ϵ from the image of the boundary of the original tile. The **ends** of an edge-tile are the points on the boundary of the edge-tile δ from a vertex of the original prototile. The remaining points on the boundary of the edge tile (when $\mathsf{d} > 2$) are uninteresting.

The edge tiles are marked by ∂ on their boundaries and labels in $\mathcal{P}(\mathcal{E}'') \cup \{\mathsf{null}\}$ in their interior, encoded by any convenient scheme, and by $[0,1] \times \mathcal{F} \cup \{\mathsf{null}\}$ on the one-dimensional facets on the out-side of the edge tile: these correspond to 1-facets on the original prototile.

3.2 Vertex tiles and vertex markings

The markings for the vertex tiles are somewhat more complicated since a single tile may carry several markings. An edge has a specific level, the level of the supertile it bounds. A vertex might be the endpoint of many different levels of

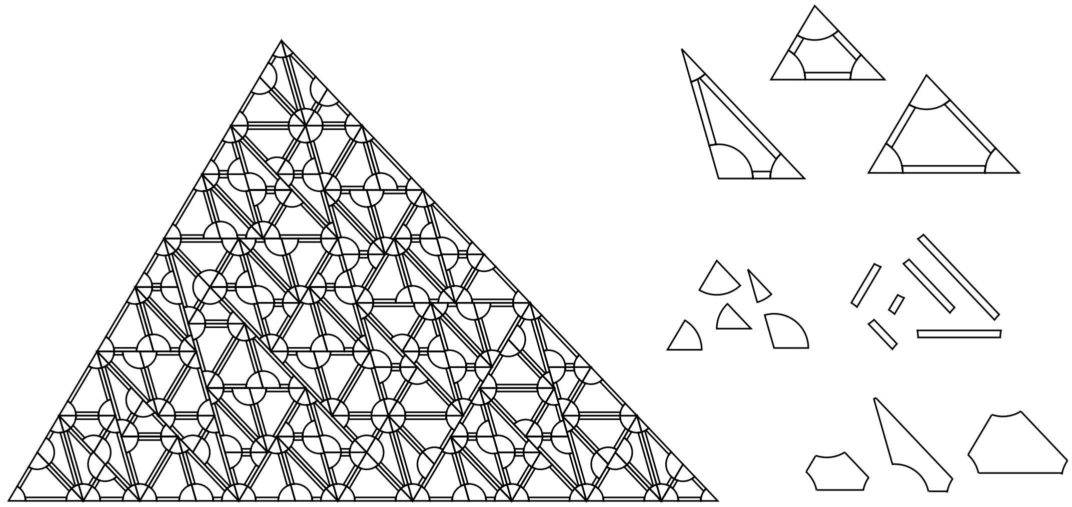

Figure 9: Small tiles, edge tiles and vertex hulls

edge and so has no specific level of its own. Instead labels in $\mathcal{V}''$ lie at specific levels and are the basis for our markings. Every vertex has finite valence and there are a finite number of vertex configurations; thus there is a bound on the number of markings required. We allow all combinations with the correct valences, satisfying certain compatibility requirements given below. The compatibility rules ensure a great degree of redundancy and in practice, one can sort out the actual combinations of marking that are required.

Recall from Section 2.2.3 that the labels $\mathcal{V}''$ each have some defined hull, an d-ball or a portion of a d-ball. The labels in $\mathcal{P}(\mathcal{V}'')$ and the hulls of their headers will form the basic vertex markings. Unions of these markings will form vertex tiles.

We will also view a vertex tile as the co-dimension 1 projection of a **stack** of these markings from a $(\mathsf{d}+1)$th dimension. The relative height of two markings corresponds to their relative height in the hierarchical structure on the tiling. Such stacks have bounded height. If we literally allow the stacking of tiles, and take the vertex markings as tiles, the number of tiles needed drops dramatically. The compatibility rules, defined shortly, are in effect vertical matching rules describing permitted stacks of markings. We will instead project these stacks down into disks or sectors, marked with many markings ranked by height, and take these as our vertex tiles.

A label $v \in \mathcal{V}''$ belongs to one of several classes. Let $v \in \mathcal{V}''(\mathsf{X}_\kappa \ldots \mathsf{A})$.

i) either v is **internal** and lies in the interior of A or is **external** and lies on the boundary of A.

ii) v arose as any or all of:

a) the endpoint of edges of relative level 1. Some of these will be among the sites serving A.

b) a site serving a lower level tile, an element of A^+.

c) the endpoint of some edge $\sigma^{(k-\kappa)}(e)$, $e \in \mathcal{E}'(X_k)$; in particular, this may be a point where the skeleton of $\sigma^k(X_k)$, $1 < k \leq \kappa$ departs $\mathcal{E}(A)$ into some element of A^+.

d) if v is external, a site for some $\sigma^k(X_k)$, $1 \leq k \leq \kappa$

e) an isolated vertex on the boundary of A

f) a terminal– that is, a meso-vertex or endo-vertex of A.

A vertex marking is a label $P \in \mathcal{P}(\mathcal{V}'')$, with the hull of its header, oriented so that the position of all edges incident to the header can be determined. The **hull of a vertex marking** is just the hull of its header. The **dark edges of a vertex marking** are the edges darkly marked in the hull of the marking. Recall that the hull of a vertex in $\mathcal{V}''$ is marked with the positions, labels in $\mathcal{E}'$ and relative levels of its dark edges, and the 1-facets of each edge have been labeled in $\mathcal{F}$ and oriented with arrows.

If the hull includes no dark edges of relative level 1 the trailer is null. If the header of P includes a dark edge of relative level 1 we regard this edge as being marked in $\mathcal{P}(\mathcal{E}'')$ with the header of this marking corresponding to the label in $\mathcal{E}'$ of this edge; we further take the trailer of this marking and the trailer of P to be the same.

Similarly, the **light edges of a vertex marking** are the lightly marked edges in its hull; the **flat sides** of a vertex marking are the flat sides of the hull.

Note that all of the internal vertices and some of the external vertices of $\sigma^n(A)$ labeled in $\mathcal{V}''(A)$ are terminals of the $\sigma^{(n-1)}(A^+)$.

One more vertex marking is possible, an **overpass**; this is simply a join for extending edges indefinitely. Such an overpass is a sector of an d-ball; each flat side must consist of a single k-plane, $0 < k < d$. Such an overpass is to be only the highest vertex marking in a stack of markings.

If the sector has two flat sides of dimensions j, k, these must share at least a common $(d - j - k)$-plane of intersection in the boundary of the sector (hence when $d = 2$, an overpass has only one flat side). This line must be marked with the orientations and labels in $\mathcal{F}$ of the 1-facets of the edges propagated by the overpass.

Along this line, an overpass propagates the boundary between higher level edges in the tiling, and also carries the orientations and labels in $\mathcal{F}$ of the 1-facets of these higher level edges. (Such a structure is necessary anyway, and helps us avoid defining tiles for every dimension from zero up through d, instead of just for dimensions 0, $(d - 1)$ and d).

3.3 Compatibility rules

We specify how we allow the vertex markings to stack:

First, the disk markings are given an order in height. The higher markings correspond to higher-level vertex hulls; they thus will be at least a large as portion of a d-ball than the lower markings. The highest marking must either be an overpass or a complete d-ball; the lowest markings must be unmarked sectors arising from hulls in $\mathcal{V}''$ meeting no edges. In between, we require that each flat side of a hull must lie exactly beneath a flat side or a edge of the next marking up; that edges and the flat sides of a hull must lie above the flat side of the hull of the next marking down; and that every light edge must be above some dark edge with the same marking in $\mathcal{E}''$.

Suppose two vertex markings $P_1, P_2 \in \mathcal{P}(\mathcal{V}'')$ contain dark edges that coincide when the stack is projected to a single tile; let the first dark edge have relative level j to P, the second relative level k to Q. We require the edges' sided arrows must be oriented the same way and have the same labels in $\mathcal{F}$ and the edges' labels in $\mathcal{E}'$ must be the same; $|j-k|$ must be less than κ. Furthermore, if both P_1, P_2 have non-null trailers, say $[..., \mathsf{Q}_{11}, \mathsf{Q}_{12}, ..., \mathsf{Q}_{1\kappa}], [..., \mathsf{Q}_{21}, \mathsf{Q}_{22}, ..., \mathsf{Q}_{2\kappa}]$ we require that $\mathsf{Q}_{2(i+j)} = \mathsf{Q}_{1(i+k)}$ for $0 \leq i \leq (\kappa - |j-k| - 1)$.

Finally, consider the vertex marking $P \in \mathcal{P}(\mathcal{V}'')$, where the header v of P arose as a site serving a tile $B \subset \sigma(A)$, $v \in \mathcal{V}''(\mathsf{X}_k...\mathsf{AB})$. If $\kappa = 1$, the trailers of the packets must be paired, as defined in Section 2.4. Otherwise, let $P = [v, \mathsf{Q}_1, ...\mathsf{Q}_\kappa]$. The hull of the marking must be a sector. Immediately above this sector must be a marking labeled P' with header $w \in \mathcal{V}''(\mathsf{X}_{(\kappa+1)}, \mathsf{X}_\kappa, ..., \mathsf{A})$ and $P = [w, \mathsf{Q}_2, ..., \mathsf{Q}_\kappa, \mathsf{Q}_{(\kappa+1)}]$. By the definitions of $\mathcal{E}''$, $\mathcal{V}''$, these two markings have at least one edge that coincide when the stack is projected to a tile; thus we are assured that w and v are appropriately matched (as they are endpoints of these edges, and the orientations and labels in $\mathcal{F}$ of the 1-facets match). The point is that a vertex tile serving as a site for a supertile must connect what appears to be the appropriate edge for the lower level skeleton to what appears to be the appropriate edge for the higher level skeleton, that the orientations and labels in $\mathcal{F}$ of the 1-facets of these edges are correct, and that the packets on these skeletons are appropriate as the packets of child and parent supertiles.

The choice of $\mathsf{Q}_{(\kappa+1)}$ is the engine of aperiodicity.

We now define **vertex tiles**: given a stack of vertex hulls satisfying the compatibility rules, project the stack to a d ball or sector. Mark all the images of the edges of hulls with ∂. In an ϵ neighborhood of these marks by ∂, mark the edge packet of the lowest dark edge above the projection.

We should consider the consequences of a hull being a terminal: First, terminals precisely arose at the vertices of the edges on the boundary of a tile, in the interior of the supertile's parent (see Section 1.4). Thus the compatibility rules ensure that above a hull that is a terminal, the appropriate edge packets of these edges are marked, and terminate (that is, these edges cannot continue through the vertex-tile). Second, note that if a terminal arose as a mesovertex, no below the hull in a stack of vertex tiles satisfying the compatibility rules is a terminal (by Lemma 1.4).

3.4 Small tiles

We defined small tiles, above, as simply prototiles with vertex and edge tiles cut away. We label these with packets in $\mathcal{Q}$, for small tiles are really just marred little supertiles. If a small tile's packet contains any non-null $\mathsf{U} \in \mathcal{U}$, $\mathsf{U} = [\mathsf{X}, \mathsf{R}]$, then the tile has a **terminator**, the label W in $\mathcal{W}$ corresponding to the vertex wire determined by U, the label R carried by U and the position and orientation of X with respect to $\mathsf{X}_1 \in \mathcal{W}$. The label v in $\mathcal{V}$ such that $\mathsf{W} \in \mathcal{W}(v)$ is implied. The terminators are the whole motivation for vertex wires.

Recall that small neighborhoods about vertices have been deleted from the original prototiles. Thus, if $\mathsf{X} = \mathsf{X}_i \in \mathsf{W}(v)$, then the terminal is marked on the boundary of a neighborhood of v_i in X; the terminal is an orientation fixing the position of v and a marking with the supertile packet R and vertex label v.

For any $\mathsf{A} \in \mathcal{T}$, there is some k such that for all n, if an edge e in $\sigma^n(\mathsf{A})$ meets a vertex $\sigma^n(v)$, $v \in \mathcal{V}(A)$, then there is an edge $\sigma^{(k-n)}(e)$ meeting $\sigma^k(v) \in \sigma^k(\mathsf{A})$; that is, after some number k of subdivisions, no further edges are incident to the vertices of A. Lightly mark $\sigma^{-k}(e) \subset \mathsf{A}$, for all $e \subset \sigma^k(\mathsf{A})$ such that e is incident to a vertex $\sigma^k(v)$. Given $\mathsf{Q} \in \mathcal{Q}(\mathsf{A})$, the edge packets are fixed for each such e; mark these on A as well. (This is unnecessary, really, but makes the matching rules more straightforward).

3.5 Matching rules $\mathcal{M}$

We define $\mathcal{T}'$ as the marked edge tiles, vertex tiles and small tiles defined above.

Recall that terminals are at endovertices or mesovertices on the boundary of a supertile these will serve to stop the propogation of the edges of the supertile's parent's skeleton. Endovertices are forced by being on the supertile's own skeleton. But mesovertices ultimately are forced by being next to a small tile marked with a terminal, at the end of a vertex-wire. Hence:

The terminal matching rule: *a terminator marked* $\mathsf{R} = \mathsf{X}_k ... \mathsf{X}_2\mathsf{X}_1$ *and* v *must be incident to a vertex tile with a terminal vertex marking* $v'' \in \mathcal{V}(\mathsf{X}_2)$ *such that* $(v \in \sigma(\mathsf{X}_2)) = \sigma(v'' \in \mathsf{X}_2)$ *and that* v'' *is aligned correctly with the orientation on the boundary of the terminator.*

We give the remaining rules. Essentially these are just that edges and vertices should fit together properly. The rules are edge to edge, and once all our labels our defined, it really is just a matter of matching markings. Of course, additional rules have already been encoded in the compatibility rules for the vertex-tile markings, the definition of well formed packets in $\mathcal{P}$, the redefinition of big tiles, the listing of the vertex hulls, and the definition of the vertex wires.

The edge matching rule: *The end of an edge tile marked* $[e, Q_1, ..., Q_k]$ *must meet a vertex tile such that with a vertex marking* $[v, Q_1, ...Q_k]$ *with e marked as a dark edge of relative level 1; the orientations and labels in* $\mathcal{F}$ *of the sided arrows must match as well. The in-side of an edge tile must meet a small tile (or big tile, defined shortly).The out-side of an edge-tile must meet a flat-side of a vertex tile or the outside of an edge-tile.* (Note we only compare markings at the ends of an edge tile!)

The tiling rule: *The tiles must cover the plane and have disjoint interiors.*

This last rule forces vertex tiles to fit in the disk-shaped holes in the small tiles. Edges to fit along the sides of the small tiles.[2]

We will give one final rule momentarily, after we formally define our labeling.

3.6 Labeling tilings in ($\mathcal{T}$, σ, $\mathcal{S}$) to obtain tilings in ($\mathcal{M}$, T')

Recall that we have defined:

$\lambda_{\mathcal{Q}} : \{\sigma^n(\mathsf{A}) \mid n \in \mathbb{N}, \mathsf{A} \in \mathcal{S}, \sigma^n(\mathsf{A}) \subset \cup \mathsf{B}_i \mathsf{A}_i\} \to \mathcal{Q}$.

$\lambda_{\mathcal{P}} : E \to \mathcal{P} \cup \{\mathsf{null}\}$ where E is the set of all points of edges of tiles in a tiling in (T, σ, $\mathcal{S}$).

$\lambda_{\mathcal{F}} : \mathcal{F} \to ([0, 1] \times \mathcal{F}) \cup \{\mathsf{null}\}$, where F is the set of points of each 1-facet of each edge of each tile in a tiling in (T, σ, $\mathcal{S}$).

These maps allow us to define explicitly a labeling λ of any tiling in ($\mathcal{T}$, σ, S): Recall we described $\lambda^{-1}(\partial)$ above. The remaining points are unambiguously in the interiors edge tiles, vertex tiles and small tiles corresponding to edges in the tiling, vertices in the tiling, and level-0 supertiles. Label each point in an edge-tile with the image under $\lambda_{\mathcal{E}}$ of the closest point in the out-side of the edge-tile (or by ∂ if this is ambiguous), its 1-facets with their images under $\lambda_{\mathcal{F}}$, each small tile with its image under $\lambda_{\mathcal{Q}}$ and markings at the curved edges derived as described in Section 3.4 (determined by the header of the label in $\mathcal{Q}$), and each vertex tile with the edge packets and orientations in $\mathcal{F}$ matching the edge tiles and small tiles to which it is incident.

Note every supertile can only be marked in finitely many ways, since each supertile can only be marked in finitely many places and their are only finitely many possible markings. So this procedure does in fact provide a labeling of ($\mathcal{T}$, σ, S).

Note, we marked infinite fault-lines with the null marking, and any infinite vertex wires carry the null supertile key.

[2] We could actually do away with the other rules entirely, for our labels can all be converted into a set of bumps and nicks that fit together only when the corresponding labels match.

As mentioned in Section 1.3, we have now defined our well-formed supertiles, configurations of tiles in $\mathcal{T}'$ that are precisely the possible labeled supertiles in tilings in $(T,\, \sigma,\, \mathcal{S})$.

3.7 Big tiles

We must finesse a certain point and define our final matching rule.

Edge tiles lying on k-level edges, $k \geq \kappa - 1$, of a tiling in $(T,\, \sigma,\, \mathcal{S})$ have been given a single marking, but lower level edge tiles might have two or more, changing at positions corresponding to skeletons of level less than κ crossing into the tiles themselves, or to sites for 0-level skeletons in the interiors of the tiles.

In some sense, this issue arises because our structures really are arbitrarily fine, but the tiles impose a certain "resolution" at which we view them. We simply impose a final matching rule resolving this issue:

The big tile matching rule:*every tile in every tiling in $(\mathcal{M},\, \mathcal{T}')$ must lie in a unique well-formed supertile of level κ*

We call these well-formed supertiles of level κ **big tiles**.

Now we have defined $\mathcal{M}$, $\mathcal{T}'$ and $(\mathcal{M},\, \mathcal{T}')$; every tiling in $(\mathcal{T},\, \sigma,\, \mathcal{S})$ can be parsed into a tiling in $(\mathcal{M},\, \mathcal{T}')$; we next attempt to show the converse.

4 The proof of the Theorem

We turn now to the space $(\mathcal{M},\, \mathcal{T}')$ of matching rule tilings. With all this work behind us, the task is relatively simple:

Recall from Section 1.3 we defined a well-formed n-level supertile to be a configuration of tiles in $\mathcal{T}'$ that is the image of a supertile in some labeling.

That is, in a well-formed supertile, we do not need to be too careful about whether this supertile lies in a labeled tiling in $(T,\, \sigma,\, \mathcal{S})$ or in a tiling in $(\mathcal{M},\, \mathcal{T}')$: our well-formed supertiles have skeletons, sites and wires clearly marked, and these structures carry packets that are consistent across the entire configuration.

We are to show that every point in the interior of a tile in an tiling in $(\mathcal{M},\, \mathcal{T}')$ lies in a unique n-level well formed supertile for each level. We would like to induct. However, this is not quite possible: in an induction, we cannot be sure of the consistency of the packets for a particular skeleton until the skeleton is completely formed; however, because a skeleton may span several levels, there may be a lag of several steps of the induction between a skeletons first appearance and the point at which it is connected. Thus, we define instead:

A configuration C of tiles in $\mathcal{T}'$ is an **almost-well-formed n-level supertile** there is a congruence $\mathsf{B} \in \mathcal{G}$ to a well-formed n-level supertile C', such that every tile in C is congruent under B to a tile in C'; and such that for any packet labeling a tile in C, all information of level n or lower is the same in the packet of the corresponding tile in C'. That is, an almost-well-formed n-level supertile differs only from a well-formed n-level supertile in that the consistency of information concerning higher level structures has not yet been checked for.

Specifically we note the $(n-1)$-level edges of the almost-well-formed supertile must all share the same supertile packet $\mathcal{Q} \in \mathsf{Q}$ because of the compatibility rules and the fact that the skeleton of the almost-well-formed supertile is connected. Because the supertile is well-formed, it is clear which edges on the boundary of the almost-well-formed supertile are meant to lie in the interior of the supertile's parent; moreover, on each such edge there is exactly one site serving the supertile, a marking at a vertex-tile v on the supertile's boundary; moreover, all vertices $\sigma^n(v)$, $v \in \mathcal{V}(\mathsf{B})$ such that v is incident to an edge in $E(\mathsf{A})$ are either meso- or endo- vertices of σ^{A} and are unambiguously identified with either a terminal or a vertex in $\mathcal{V}''(\mathsf{A})$.

In figure 10 we see a schematic of an almost well-formed supertile. Note that the skeleton of the supertile itself, a packet in $\mathcal{Q}$ on the skeleton, the sites for daughter supertiles and vertex wires to the meso-vertices are fixed. Terminators are in fixed orientations and positions. In addition, any edges for higher level supertiles that pass through our almost-well-formed supertile are determined, although the packet in $\mathcal{Q}$ on the skeleton is not fixed. Similarly, vertex wires to the epi-vertices may or may not be present.

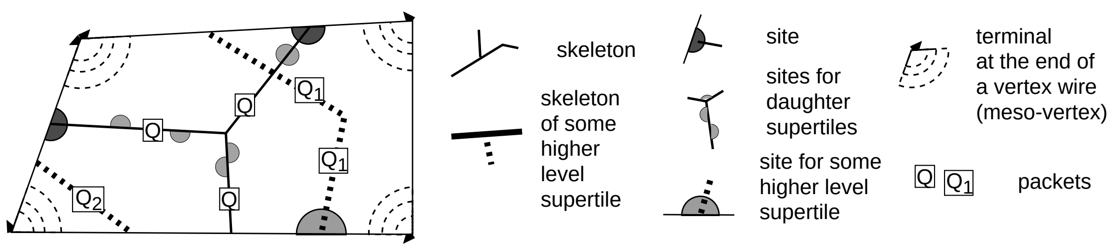

Figure 10: Structures in an almost-well-formed supertile

Since our big tiles are themselves almost-well-formed supertiles of level κ, the following proposition, with Lemma 4.2, completes an inductive proof of the Theorem.

Proposition 4.1 *If any almost-well-formed supertile of level j, $j \leq n$ in any tiling in $(\mathcal{M}, \mathcal{T}')$ lies in an almost well-formed supertile of level n, every almost well formed supertile of level j, $j \leq n$ in any tiling in $(\mathcal{M}, \mathcal{T}')$ lies in an almost well- formed supertile of level $n+1$.*

Proof We proceed by induction and may assume that $n > \kappa$ (which we may, by the Big Tile Rule) and that every tile lies in an almost-well-formed supertile of each level $k < n$.

But we have gone to a great deal of trouble already: the work is already done!

(1) Consider an almost well-formed n-level supertile S, congruent to $\sigma(\mathsf{X}_1)$ for some $\mathsf{X}_1 \in \mathcal{S}$, with supertile packet $\mathsf{Q} = [\mathsf{X}_\kappa ... \mathsf{X}_2\mathsf{X}_1, ...] \in \mathcal{Q}$. We must show our almost-well-formed supertile lies in an almost-well-formed supertile S' congruent to $\sigma^{(n+1)}(\mathsf{X}_2)$. That is, we must check that our original supertile and its siblings form the correct configuration and that the markings on the skeleton and vertex wires of $\sigma^{(n+1)}(\mathsf{X}_2)$ are correct, up to structures of level $n+1$. In particular, the skeleton of $\sigma^{(n+1)}(\mathsf{X}_2)$ must carry a supertile packet Q' paired with Q (Section 2.4). Our starting point is suggested in (1) in figure 11; our goal is (6).

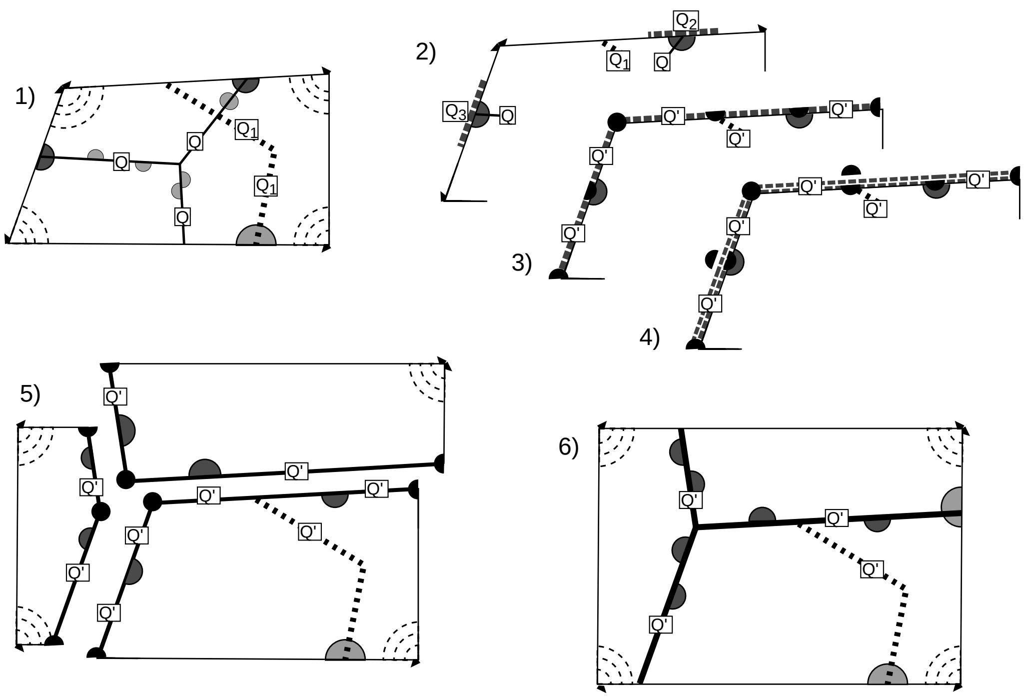

Figure 11: The proof of the main theorem

(2) Since S is almost-well-formed , we can be sure that the sites (Section 2.1.2) $\sigma^{n-1}(\mathsf{Z}(\mathsf{X}_1))$ are correctly positioned, oriented and labeled, up to structures of level n, on the boundary of S. In particular, at each site $\sigma^{n-1}(z)$, $\mathsf{Z}(\mathsf{X}_1)$, the compatibility rules (Section 3.3) ensure that immediately above the site's own vertex hull is a vertex hull in $\mathcal{V}''(\mathsf{X}_2)$ (Section 2.2.3), with edges marked

with supertile packet Q' where Q, Q' are paired (Section 2.4). Moreover, at this site the edges corresponding to the n-level edges in the skeleton of $\sigma^{n+1}(\mathsf{X}_2)$ are all correctly labeled and oriented (Section 2.2.3). Let the header of Q' be the supertile key $\mathsf{X}_{\kappa+1} \ldots \mathsf{X}_2$.

Now the tiling rule and the edge matching rule (Section 3.5) force edge tiles to be be placed next to the given site, with edge markings indicating the tiles are in the skeleton of E_{X_2}, on edges $e \in \mathcal{E}''(\mathsf{X}_{\kappa+1} \ldots \mathsf{X}_2)$ incident to the site $z \in \mathsf{Z}_(\mathsf{X}_1)$, carrying supertile packet Q', oriented correctly with respect to S.

By the edge rules, each of these edge tiles must propogate a series of edge tiles and overpasses, all with the same edge key and orientation, following an edge of S.

Note however, that at different sites, at the moment we have no guarentee that all the supertile packets of the parent $\sigma^{n+1}(\mathsf{X}_2)$ are equivalent.

(3) By the edge rules, these series of edge tiles and overpasses must continue until the appearance of a vertex in $\mathcal{V}''(\mathsf{X}_{\kappa+1} \ldots \mathsf{X}_2)$ (Section 2.2.3). If this hull lies at a terminal, the series must completely stop; if this hull lies at some other vertex in the skeleton $\sigma^n(E_{\mathsf{X}_2})$, the edge continues, but the header of the edge packet changes (the edge tiles are now on a new edge of $\sigma^n(\mathcal{E}''(\mathsf{X}_2))$ but the trailer remains the same.

But S is an almost-well-formed supertile, and so has vertex tiles with hulls in $\mathcal{V}''(\mathsf{X}_2)$ — identifying vertices in $\sigma^n(\mathcal{V}(\mathsf{X}_1))$ (Section 2.1.1)— correctly positioned, oriented, and labeled up to level n. In particular, along $\sigma^n(e)$, terminals are present only at the endpoints of $\sigma^{(n)}(e)$ Thus the series of edge tiles and overpasses along $\sigma^n(e)$ *must* stop, and *can only* stop at the endpoints of $\sigma^n(e)$.

Moreover if other vertices in $\sigma^n(\mathcal{V}''(x_{\kappa+1} \ldots \mathsf{X}_2)$ are meant to be present in $\sigma^n(e)$ these vertices have been fixed already by lower level structures in S. Thus the edge packets along $\sigma^n(e)$ are consistent.

(4) Because the original tiling is sibling edge-to-edge, our terminals at the end of $\sigma^n(e)$ must be terminals for the edges of neighboring supertiles; moreover, these edges must convey the same trailer of the packet in $\mathcal{P}$. However, the nature of these neighboring supertiles, and their precise alignment is not yet determined. But these terminals do ensure that edge tiles must propogate back along $\sigma(-e)$. The edge packets on these tiles must convey the same supertile packet Q' as our initial edge tiles; moreover, every vertex on $\sigma^n(-e)$ must be present or the matching rules would be violated somewhere on $\sigma^n(-e)$.

(5) In particular, we must eventually come across the site lying to the other side of the edge; this site must be adjacent to some big tile; this big tile must lie in some well-formed n-level supertile. (6) Since this supertile must share the same vertices as $\sigma^n(\mathsf{X}_1)$, its position and nature are fixed: it is indeed the appropriate sibling of $\sigma^n(\mathsf{X}_1)$, in the correct position.

Now note that information concerning level higher than n supertiles in the packet in the tiles in $\sigma^n(\mathsf{A})$ and $\sigma(\mathsf{C})$ must be correlated by this edge. And so we can walk our way about, showing each sibling in turn is in its place.

Finally, because the skeleton E_{X_2} is connected, we have that the supertile packet is consistent across all of $\sigma^{n+1}(\mathsf{X}_2)$; moreover this packet ensures all structures are correctly placed and oriented. Thus, our original almost-well-formed supertile of level n lies in an almost well-formed supertile of level $n+1$. □

The following lemma completes the proof of the theorem.

Lemma 4.2 *If every point in the interior of every tiling in ($\mathcal{M}$,$\mathcal{T}'$) lies in a unique almost-well-formed supertile of each level $n \in \mathbb{N}$ then every point in the interior of every tiling in ($\mathcal{M}$,$\mathcal{T}'$) lies in a unique well-formed supertile of every level n.*

Proof This is simply because any almost well-formed supertile of level n in an almost-well-formed supertile of level greater than $n+\kappa$ is in fact a well-formed supertile. □

We used the condition in the statement of the theorem very strongly: we needed a mechanism– vertex wires– to fix the position of the vertices of $\sigma(\mathsf{A})$ and thus keep edges from propagating beyond their intended borders. The vertex wires required vertices to be hereditary. Secondly, to fix the position of sibling supertiles relative to some initial supertile, we need some point in each we can say they share– they are forced to share vertices.

Other mechanisms can be devised, exploiting other conditions one could make. However it is not clear if an example exists in which the relative positions of sibling tiles cannot be fixed at all.

Note that we have specifically shown that every tile in a tiling in ($\mathcal{M}$, $\mathcal{T}$) lies in arbitrarily large well-formed supertiles. This is the definition we have taken for enforcement. Note though, that if the tiling in ($\mathcal{M}$, $\mathcal{T}'$) has an infinite fault, we have no control over the edge-label propagated down the fault; similarly a vertex wire might serve no highest level supertile. We also have no control over slippage along such a fault. In the vertex to vertex case, we can stiffen this structure up considerably.

In [17, 19] and elsewhere, it is noted that the correspondence between the tilings in ($\mathcal{M}$, $\mathcal{T}'$) and tilings in ($\mathcal{T}$, σ, $\mathcal{S}$) is one-to-one, except on a set of measure zero in any translation invariant probability measure on ($\mathcal{T}$, σ, $\mathcal{S}$).

We conclude with a final definition, lemma and corollary.

A matching rule tiling ($\mathcal{M}$, $\mathcal{T}'$)$\neq \emptyset$ is **aperiodic** if no tiling in ($\mathcal{M}$, $\mathcal{T}'$) is invariant under some infinite cyclic group of isometries.

Lemma 4.3 *Any matching rule tiling $(\mathcal{M}, \mathcal{T}')$ constructed in the proof of our Main Theorem is aperiodic.*

Proof In fact, the proof of the Proposition yields that every tile in a tiling in $(\mathcal{M}, \mathcal{T}')$ lies in a unique well-formed supertile, for every edge tile must play a clearly identifiable role in the skeletons of well-formed supertiles in the hierarchy, and so the well-formed supertiles themselves are clearly identifiable. Every element B of any infinite cyclic group of isometries in $\mathbb{E}^{\mathsf{d}}$ acting on a tiling in $(\mathcal{M}, \mathcal{T}')$ is fixed point free. There is a well-formed supertile $\sigma^n(\mathsf{A})$ in the tiling larger than the distance between a point and its image under the isometry, so the tiling cannot be invariant under B (or any tile $\mathsf{C} \subset \sigma^n(\mathsf{A})$ would also lie in the well-formed supertile $\mathsf{B}\sigma^n(\mathsf{A})$). □

Note in particular the above lemma holds even if there are periodic tilings in $(\mathcal{T}, \sigma, S)$; this is because we are specifically enforcing the hierarchy inherent in the substitution tiling, which is not periodic (See [7]).

With the observation that there are infinitely many distinct substitution tilings (cf [7]), we have:

Corollary 4.4 *There are infinitely many aperiodic, hierarchical matching rule tilings.*

5 A simple example

We illustrate the construction with a variation on one of Danzer's tilings [5]; this example has the distinction of being one of the very simplest possible examples with which to work. (This is because mostly because $\mathcal{S}$ is small, the skeletons are exceedingly simple, and $\kappa = 1$.)

We take the substitution illustrated below. In figure 12, $\mathcal{T} = \{\mathsf{X}, \mathsf{Y}\}$, $\mathcal{S} = \{\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}\}$, σ, $\sigma^6(\mathsf{X})$, $\sigma^6(\mathsf{Y})$ are illustrated.

We next begin defining structures for the construction (figure 13). As in Section 2.1.1 we define
$\mathcal{V}(\mathsf{A}) = \{1\mathsf{A}, 2\mathsf{A}, 3\mathsf{A}\}$,
$\mathcal{V}(\mathsf{B}) = \{4\mathsf{B}, 5\mathsf{B}, 6\mathsf{B}\}$,
$\mathcal{V}(\mathsf{C}) = \{1\mathsf{C}, 2\mathsf{C}, 3\mathsf{C}\}$,
$\mathcal{V}(\mathsf{D}) = \{4\mathsf{D}, 5\mathsf{D}, 6\mathsf{D}\}$ and $\mathcal{V} = \mathcal{V}(\mathsf{A}) \cup \mathcal{V}(\mathsf{B}) \cup \mathcal{V}(\mathsf{C}) \cup \mathcal{V}(\mathsf{D})$.
Note that 2A, 6B, 4D, and 1C are mesovertices; 3A, 4B, 3C and 5D are endovertices and the rest are epivertices.

We next define: $\mathcal{E}(\mathsf{X}) = \{+\mathsf{i}, -\mathsf{i}\} \subset \sigma(\mathsf{X})$, $\mathcal{E}(\mathsf{Y}) = \{+\mathsf{ii}, -\mathsf{ii}\} \subset \sigma(\mathsf{Y})$ and $\mathcal{E} = \{+\mathsf{i}, -\mathsf{i}, +\mathsf{ii}, -\mathsf{ii}\}$.

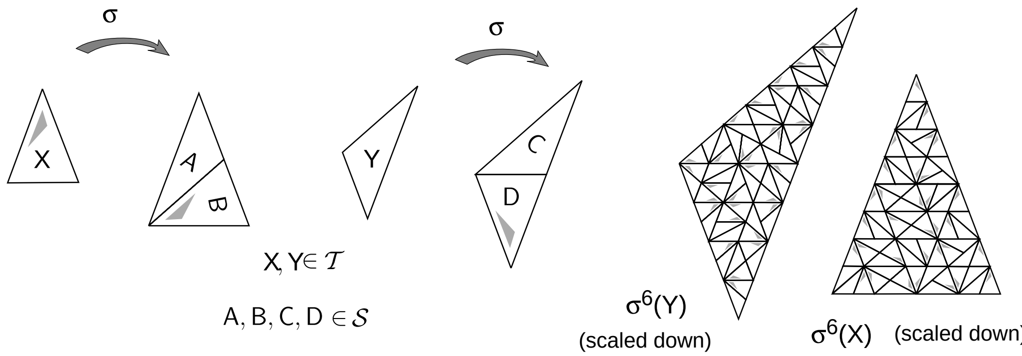

Figure 12: $\mathcal{T}$, σ and $\mathcal{S}$

We turn to sites (figure 14, Section 2.1.2). Each edge meets a suitable site after only one inflation. Moreover, the edges in $\mathcal{E}(\mathsf{X})$, $\mathcal{E}(\mathsf{Y})$ are connected. So we are fortunate enough to have $\kappa = 1$ (Section 2.1.3) (Again, this is unexceptional– most well known examples have $\kappa = 1$). We take

$\mathcal{Z}(\mathsf{A}) = \{3\mathsf{A} = \mathsf{z}(-\mathsf{i})\}$,
$\mathcal{Z}(\mathsf{B}) = \{4\mathsf{B} = \mathsf{z}(+\mathsf{i})\}$,
$\mathcal{Z}(\mathsf{C}) = \{3\mathsf{C} = \mathsf{z}(-\mathsf{ii})\}$, and
$\mathcal{Z}(\mathsf{D}) = \{5\mathsf{D} = \mathsf{z}(+\mathsf{ii})\}$.

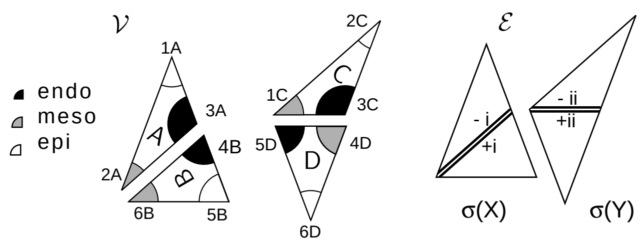

Figure 13: $\mathcal{V}$, $\mathcal{E}$

Our skeletons (Section 2.1.4) will just be the edges $\mathcal{E}$, but within elements of $\sigma(\mathcal{S})$. We define the vertices and edges of the skeletons:

$\mathcal{V}'(\mathsf{A}) = \{\mathsf{IA}, \mathsf{IIA}, \mathsf{VIA}, \mathsf{VIIA}\} \subset \sigma(\mathsf{A})$,
$\mathcal{V}'(\mathsf{B}) = \{\mathsf{IIIB}, \mathsf{IVB}, \mathsf{VB}, \mathsf{VIIIB}\} \subset \sigma(\mathsf{B})$,
$\mathcal{V}'(\mathsf{C}) = \{\mathsf{IC}, \mathsf{IIC}, \mathsf{VIC}, \mathsf{VIIC}\} \subset \sigma(\mathsf{C})$,
$\mathcal{V}'(\mathsf{D}) = \{\mathsf{IIID}, \mathsf{IVD}, \mathsf{VD}, \mathsf{VIIID}\} \subset \sigma(\mathsf{D})$,
and $\mathcal{V}' = \mathcal{V}'(\mathsf{A}) \cup \mathcal{V}'(\mathsf{B}) \cup \mathcal{V}'(\mathsf{C}) \cup \mathcal{V}'(\mathsf{D})$.

$\mathcal{E}'(\mathsf{A}) = \{\mathsf{iiA}\} \subset \sigma(\mathsf{A})$,
$\mathcal{E}'(\mathsf{B}) = \{\mathsf{iB}\} \subset \sigma(\mathsf{B})$,
$\mathcal{E}'(\mathsf{C}) = \{\mathsf{iiC}\} \subset \sigma(\mathsf{C})$,

$\mathcal{E}'(\mathsf{D}) = \{\mathsf{iD}\} \subset \sigma(\mathsf{C})$, and $\mathcal{E}' = \mathcal{E}'(\mathsf{A}) \cup \mathcal{E}'(\mathsf{B}) \cup \mathcal{E}'(\mathsf{C}) \cup \mathcal{E}'(\mathsf{D})$.

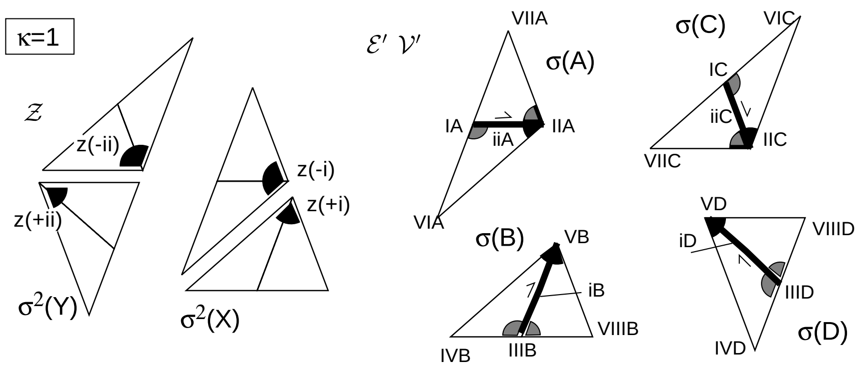

Figure 14: $\mathcal{Z}$, $\mathcal{V}'$, and $\mathcal{E}'$

Since $\kappa = 1$, $\mathcal{R} = \mathcal{S}$ (Section 2.2.1), and the elements of $\mathcal{E}''$ and $\mathcal{V}''$ are just those of $\mathcal{E}'$ and $\mathcal{V}'$. However, the elements of $\mathcal{V}''$ are identified with their hulls (Section 2.2.2), shown in figure 15. Recall the hulls are marked with **dark edges** and **light edges**. The dark edges are black; those of relative level one are thick; those of relative level zero are thin. The light edges are gray.

At this point we orient the one-facets of our edges with arrows; to collapse the notation, "side" our arrows: for example, the hook on the head of each arrow on a pair of edges $+e$, $-e$ points to $-e$; we will consequently compound $\pm e$ for each $e \in \mathcal{E}''$. (These arrows first appear in figure 14 since $\mathcal{E}' = \mathcal{E}''$).

We next define vertex wires for the mesovertices 2A, 6B, 1C and 4D (figure 16 and Section 2.3). Thus
$\mathsf{W}(2\mathsf{A}) = [\mathsf{A}_{2\mathsf{A}} \bullet \mathsf{C}_{2\mathsf{A}}]$,
$\mathsf{W}(6\mathsf{B}) = [\mathsf{B}_{6\mathsf{B}} \bullet \mathsf{A}_{6\mathsf{B}}, \mathsf{D}_{6\mathsf{B}}]$,
$\mathsf{W}(1\mathsf{C}) = [\mathsf{C}_{1\mathsf{C}} \bullet \mathsf{D}_{1\mathsf{C}}, \mathsf{A}_{1\mathsf{C}}]$,
$\mathsf{W}(4\mathsf{D}) = [\mathsf{D}_{4\mathsf{D}} \bullet \mathsf{B}_{5\mathsf{B}}]$.

Recall that the elements of $\mathsf{W}(v)$ for any mesovertex v are elements of $\mathcal{S}$, but with additional information– the orientation of the tile and the mesovertex served.
$l(2\mathsf{A}) = l(6\mathsf{B}) = l(1\mathsf{C}) = l(4\mathsf{D}) = 0$;
$m(2\mathsf{A}) = m(4\mathsf{D}) = 1$, and $m(6\mathsf{B}) = m(1\mathsf{C}) = 2$.

Recall the definition of v_n given mesovertex v. Thus
$2\mathsf{A}_0 = 2\mathsf{A}$ and $2\mathsf{A}_1 = 2\mathsf{C}_{2\mathsf{A}}$,
$6\mathsf{B}_0 = 6\mathsf{B}$, $6\mathsf{B}_1 = 1\mathsf{A}_{6\mathsf{B}}$ and $6\mathsf{B}_2 = 6\mathsf{D}_{6\mathsf{B}}$,
$1\mathsf{C}_0 = 1\mathsf{C}$, $1\mathsf{C}_1 = 6\mathsf{D}_{1\mathsf{C}}$ and $1\mathsf{C}_2 = 1\mathsf{A}_{1\mathsf{C}}$, and

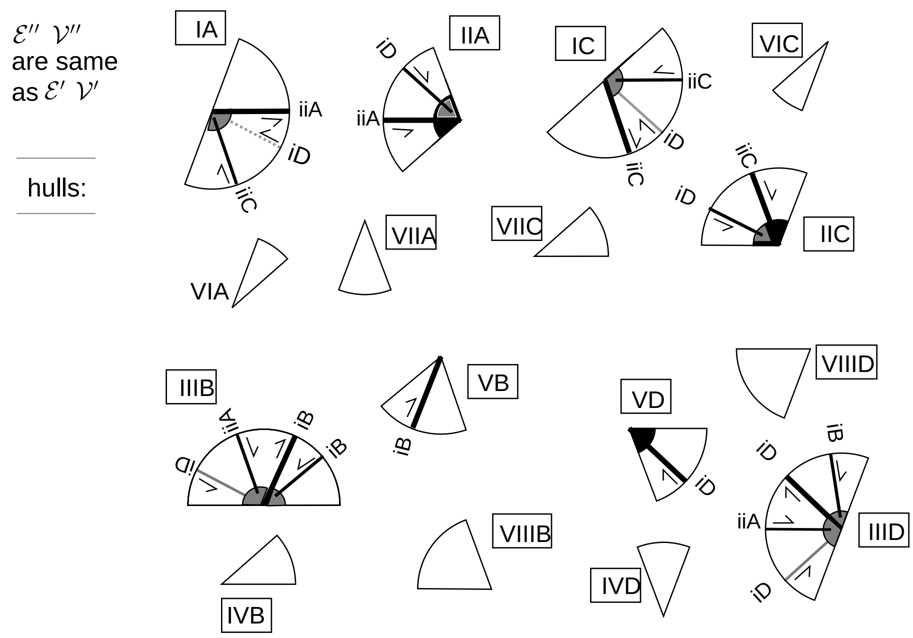

Figure 15: Vertex hulls for elements of $\mathcal{V}''$

$4\mathsf{D}_0 = 4\mathsf{D}$ and $4\mathsf{D}_1 = 5\mathsf{B}_{4\mathsf{D}}$.

Thus, for epivertices 1A, 5B, 6D and 2C we define $\mathcal{U}$
$\mathcal{U}(1\mathsf{A}) = \{(\mathcal{W}(6\mathsf{B})_1 = \mathsf{A}_{6\mathsf{B}}, \mathsf{B}), (\mathcal{W}(1\mathsf{C})_2 = \mathsf{A}_{1\mathsf{C}}, \mathsf{C})\}$,
$\mathcal{U}(6\mathsf{D}) = \{(\mathcal{W}(6\mathsf{B})_2 = \mathsf{D}_{6\mathsf{B}}, \mathsf{B}), (\mathcal{W}1\mathsf{C})_1 = \mathsf{D}_{1\mathsf{C}}, \mathsf{C})\}$,
$\mathcal{U}(2\mathsf{C}) = \{(\mathcal{W}(2\mathsf{A})_1 = \mathsf{C}_{2\mathsf{A}}, \mathsf{A})\}$, and
$\mathcal{U}(5\mathsf{B}) = \{(\mathcal{W}(4\mathsf{D})_1 = \mathsf{B}_{4\mathsf{D}}, \mathsf{D})\}$.

We next turn to the packets any given supertile may have to carry (Section 2.4). Since our wires are short, our epivertices few, and $k = 1$, these are blessedly few. We'll collapse the notation as we go, reusing our labels in $\mathcal{W}$ as names of supertile packets.
$\mathcal{Q}(\mathsf{A}) = \{[\mathsf{A}, (\mathcal{W}(6\mathsf{B})_1 = \mathsf{A}_{6\mathsf{B}}, \mathsf{B})] = \mathsf{A}_{6\mathsf{B}}, [\mathsf{A}, (\mathcal{W}(1\mathsf{C})_2 = \mathsf{A}_{1\mathsf{C}}, \mathsf{C})] = \mathsf{A}_{1\mathsf{C}}\}$,
$\mathcal{Q}(\mathsf{B}) = \{[\mathsf{B}, (\mathcal{W}(4\mathsf{D})_1 = \mathsf{B}_{4\mathsf{D}}, \mathsf{D})] = \mathsf{B}_{4\mathsf{D}}\}$,
$\mathcal{Q}(\mathsf{C}) = \{[\mathsf{C}, (\mathcal{W}(2\mathsf{A})_1 = \mathsf{C}_{2\mathsf{A}}, \mathsf{A})] = \mathsf{C}_{2\mathsf{A}}\}$,
$\mathcal{Q}(\mathsf{D}) = \{[\mathsf{D}, (\mathcal{W}(6\mathsf{B})_2 = \mathsf{D}_{6\mathsf{B}}, \mathsf{B})] = \mathsf{D}_{6\mathsf{B}}, [\mathsf{D}, (\mathcal{W}(1\mathsf{C})_1 = \mathsf{D}_{1\mathsf{C}}, \mathsf{C})] = \mathsf{D}_{1\mathsf{C}}\}$.

Next, we give the edge packets. Again, one is relieved that the example is simple. We provide reasonable names for the packets as we go. Possible edge packets are:
$\mathcal{P}(\mathcal{E}'') = \{[\mathsf{iiA}, \mathsf{A}_{1\mathsf{C}}] = \mathsf{iiA}_{1\mathsf{C}}$,
$[\mathsf{iiA}, \mathsf{A}_{6\mathsf{B}}] = \mathsf{iiA}_{6\mathsf{B}}$,
$[\mathsf{iB}, \mathsf{B}_{4\mathsf{D}}] = \mathsf{iB}_{4\mathsf{D}}$

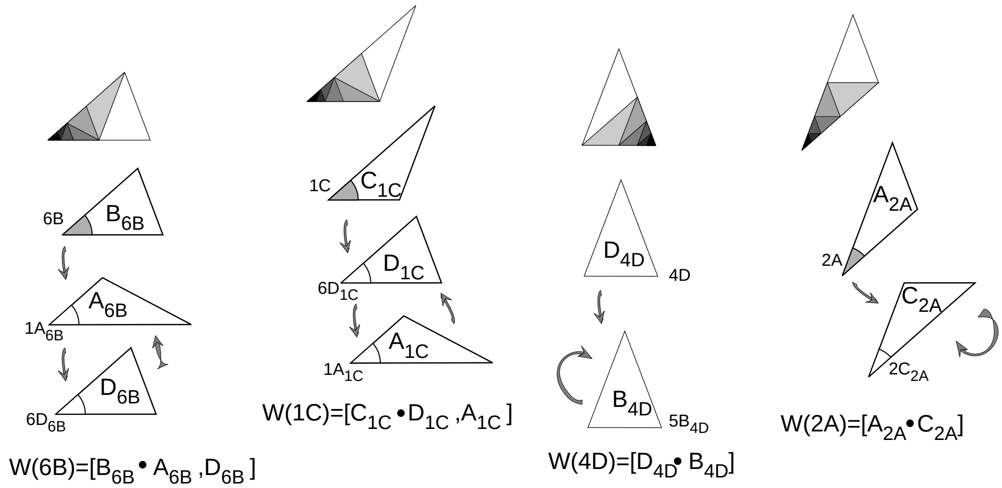

Figure 16: Vertex wires for the mesovertices

$[\mathsf{iiC}, \mathsf{C_{2A}}] = \mathsf{iiC_{2A}}$,
$[\mathsf{iD}, \mathsf{D_{6B}}] = \mathsf{iD_{6B}}, [\mathsf{iD}, \mathsf{D_{1C}}] = \mathsf{iD_{1C}}\}$.

Similarly, we define vertex packets. Recall we do not bother defining these for vertices unless they meet the skeleton of the tile in $\mathcal{S}$ to which they belong. Thus, we do not define packets for VIA, VIIA,VIIIB, IVB, VIC, VIIC, VIIID, or IVD.

The vertex packets are:
$\mathcal{P}(\mathcal{V}'') = \{[\mathsf{IA}, \mathsf{A_{1C}}] = \mathsf{IA_{1C}}, [\mathsf{IIA}, \mathsf{A_{1C}}] = \mathsf{IIA_{1C}}$,
$[\mathsf{IA}, \mathsf{A_{6B}}] = \mathsf{IA_{6B}}, [\mathsf{IIA}, \mathsf{A_{6B}}] = \mathsf{IIA_{6B}}$,
$[\mathsf{IIIB}, \mathsf{B_{4D}}] = \mathsf{IIIB_{4D}}, [\mathsf{VB}, \mathsf{B_{4D}}] = \mathsf{VB_{4D}}$,
$[\mathsf{IC}, \mathsf{C_{2A}}] = \mathsf{IC_{2A}}, [\mathsf{IIC}, \mathsf{C_{2A}}] = \mathsf{IIC_{2A}}$,
$[\mathsf{IIID}, \mathsf{D_{6B}}] = \mathsf{IIID_{6B}}, [\mathsf{VD}, \mathsf{D_{6B}}] = \mathsf{VD_{6B}}$,
$[\mathsf{IIID}, \mathsf{D_{1C}}] = \mathsf{IIID_{1C}}, [\mathsf{VD}, \mathsf{D_{1C}}] = \mathsf{VD_{1C}}\}$.

We turn to the tiles. First, in figure 17, the unmarked tiles themselves are shown (Section 3). The vertex tiles will be made of the various sectors illustrated; they will be half disks when fully assembled (because no element of V lies in the interior of a $\sigma(\mathsf{X})$ or $\sigma(\mathsf{Y})$ none of our vertex tiles will be full disks). Following this, the small tiles, marked with supertile packets and indications of the interior edges, are shown (figure 18). (In our illustration, we can take the edge packets in the interior edges of the small tiles, and the terminals at the epivertices as understood: for example, tile $\mathsf{A_{IC}}$ has a terminal marking at its upper vertex: this terminal must match with hull $\mathsf{VIIA} \in \mathcal{V}''(\mathsf{A})$, corresponding to vertex $\mathsf{1A} \in \mathcal{V}(A)$ marked with supertile key C.

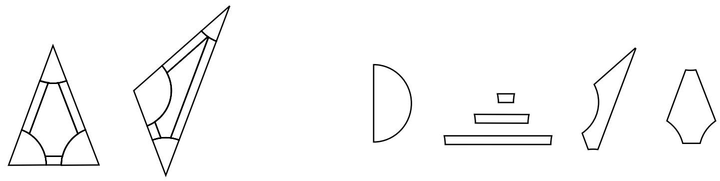

Figure 17: The unmarked tiles

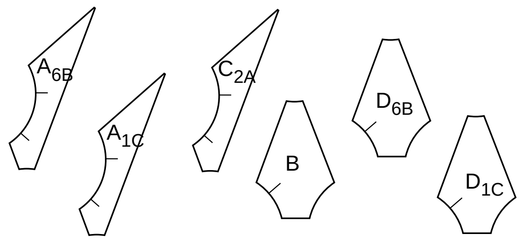

Figure 18: The small tiles marked with supertile packets

In the next illustration, the edge tiles are marked (figure 19). Note that we allow orientation reversing isometries of our edge-tiles. If we regard the edge packets and orientations as residing on little tiles we implant into our edge tiles, we can reduce the number of tiles needed here from $9 \times 2 \times 3 = 54$ to $9 + 1 + 3 = 13$.

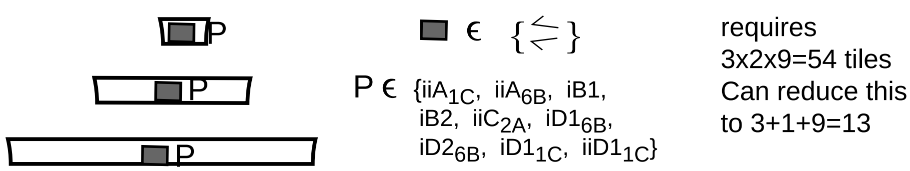

Figure 19: Marked edge tiles

The vertex hulls are next assembled into actual vertex tiles. In this example, there are not very many possibilities allowed by the compatibility rules (Section 3.3). In particular, there are only six basic kinds of stack allowed; each can be capped off by any of the 9×4 overpass decorations (each corresponding to an edge packet) giving two hundred sixteen vertex tiles. We can reduce this to $6 + 9 + 1 = 16$ tiles by breaking the tiles up into independent pieces as shown in figure 20.

Note that we thus have a grand total of six small tiles, fifty-four edge tiles, and two hundred sixteen vertex tiles, for a total of 276 tiles altogether, or if we break the tiles up further, six small tiles, three blank edge-tiles, nine edge-packet tiles, one tiles indicating orientation of the edges, and six vertex tiles, for a total of 25 tiles. With imagination we might reduce this number even further.

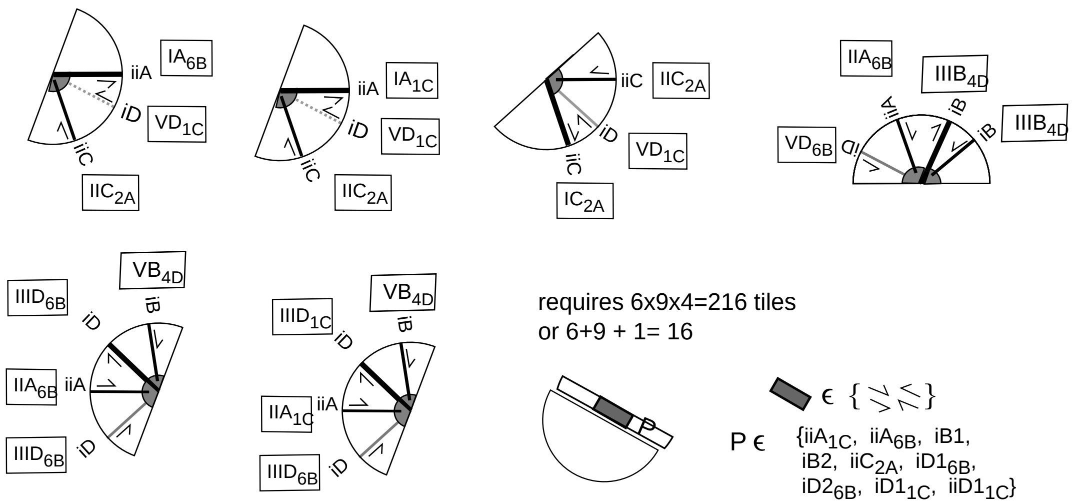

Figure 20: Marked vertex tiles, overpasses

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