# Addressing in substitution tilings

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**Introduction**

Substitution tilings have been discussed now for at least twenty-five years, initially motivated by the construction of hierarchical non-periodic structures in the Euclidean plane [**?**, **?**, **?**, **?**]. Aperiodic sets of tiles were often created by forcing these structures to emerge. Recently, this line was more or less completed, with the demonstration that (essentially) every substitution tiling gives rise to an aperiodic set of tiles [**?**].

Thurston and then Kenyon have given remarkable characterizations of self-similar tilings, a very closely related notion, in algebraic terms [**?**, **?**]. The dynamics of a species of substitution tilings under the substitution map has been extensively studied [**?**, **?**, **?**, **?**, **?**].

However, to a large degree, a great deal of detailed, local structure appears to have been overlooked. The approach we outline here is to view the tilings in a substitution species as realizations of some algorithmicly derived language—in much the same way one can view a group's elements as strings of symbols representing its generators, modulo various relations.

This vague philosophical statement may become clearer through a quick discussion of how substitution tilings have often been viewed and defined, compared to our approach. The real utility of addressing, however, is that we can provide detailed and explicit descriptions of structures in substitution tilings.

Figure 1: A typical substitution

We begin with a rough heuristic description of a substitution tiling (formal definitions will follow below): One starts in affine space $\mathbb{R}^n$ with a finite collection $T$ of prototiles,[1], a group $G$ of isometries to move prototiles about, an

[1]In the most general context, we need assume little about the topology or geometry of the

expanding affine map $\sigma$ and a map $\sigma'$ from prototiles to tilings– substitution rules. These substitution rules specify how to map each prototile to a tiling of its image under $\sigma$.

For example, in figure **??** we have a single prototile– the $L$-tile or chair tile [**?**, **?**]– residing in the plane. Our inflation $\sigma$ simply scales all distances by a factor of 2. Our replacement rule takes the inflated $L$-tile and replaces it with four images of the $L$-tile as shown. On the right is a portion of a tiling that clearly, in some sense, deserves to be called a "substitution tiling", derived from this substitution.

A substitution tiling given by $T$,$\sigma$ and $\sigma'$ is often described as the result of repeatedly iterating the two steps "inflate" and then "subdivide". This certainly is intuitively clear, and one can easily define supertiles— the result of finitely many iterations. Two points must be made, however. First, unless one takes some care, this process has no limit; generically, patches of tiles are not stable under inflation. That is, although a given patch's image under inflation typically often contains a congruent copy of the patch, it very rarely contains the patch itself. We precisely characterize this in Section **??**.

Second, being able to tile supertiles of arbitrary size does imply the plane can be tiled, but one has to appeal to a somewhat opaque argument (eg. Theorem 3.8.1 of [**?**]). Moreover, from this description, it is not at all clear how the entire species of tilings arising from $T$,$\sigma$ and $\sigma'$ should be defined.

The usual formal description of the species is quite elegant, but reveals its structure reluctantly. A tiling $\tau$ of $X$ with images of prototiles in $T$ is an element of the substitution species $\Sigma(T, \sigma')$ if and only if every bounded set in $\tau$ is the image of a bounded subset of some supertile, as described above. This certainly is well defined, but even simple properties– such as not being vacant— of these species often require involved proof.

We proceed as follows. Instead of inflating tiles and expecting a limit, we simply construct "addresses"— essentially the position of tile with respect to a hierarchy of supertiles containing it. We begin by identifying a set of symbols $S$ not with the prototiles, but with the tiles in the images of the prototiles under substitution. The elements of $S$ can be thought of as the prototiles with additional information attached— the identity and relative position of the prototiles "parent".

Thus, we can build infinitely large configurations of tiles as follows: begin with a tile, specify its parent, specify that supertile's parent, etc. Eventually one has an infinite hierarchy of supertiles— an infinite-level supertile.

tiles, the nature of the geometry of the space,etc. Some of these issues are discussed in Section **??**

For example, in Figure **??**, $S$ has four elements, denoted $\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}$. At right, a portion of an infinite-level supertile is being built up; a tile is assigned some address $\ldots\mathsf{cbda}\bullet$ indicating that the tile itself is in position $\mathsf{a}$ with respect to its parent; the parent is in position $\mathsf{d}$ with respect to the tile's grandparent; etc. (This is unambiguous once the specific isometries used to assemble the diagram on the left of Figure **??** are specified— here we use only translations and rotations). Note that the limit of this description— an infinite-level supertile— is perfectly well-defined.

Figure 2: Addressing supertiles

Note that the infinite-level supertile can be described from any initial tile within it— thus these supertiles really correspond to equivalence classes of addresses (essentially, two addresses give rise to the same infinite-level supertile if and only if they agree in all but finitely many digits), and addresses of the sort in the example above correspond to positions of tiles within infinite-level supertiles. In Figure **??**, addresses are given for a number of tiles in the supertiles shown. The full addresses of these tiles will all agree to the left of the fourth digit shown.

Several points need be made: First, infinite-level supertiles are inextricably bound to these addresses. Consequently, the construction of the infinite-level supertiles is explicit and the set of infinite-level supertiles can be explicitly understood, at least to roughly the same extent as the real numbers can be explicitly understood.

Second, note that the addresses are algorithmically generated, and in fact the substitution graph (Section **??**) is the finite state automaton that generates the addresses of infinite-level supertiles.

Third, if an infinite-level supertile covers $\mathbb{R}^n$, it is a substitution tiling, using the standard definition given above. Moreover— though this takes detailed proof— every substitution tiling can be assembled from infinite-level supertiles. In fact, under any $G$-invariant probability measure on a given species of substitution tilings, the set of substitution tilings that consist of a single infinite-level supertile has measure 1! (Section **??**; also: [**?**, **?**, **?**])

Fourth, we can construct an explicit relation that exactly describes if two infinite-supertiles belong together in some substitution tiling. Up to fairly sharp conditions, we can produce two automata: one that can check whether two addresses are related in this fashion, and one that produces all addresses of infinite-level supertiles incident to the infinite-level supertile given by a given address.

There is a strong analogy to the real numbers themselves. We have no trouble checking that the numbers 9999 and 10000 are adjacent. In similar fashion, we can construct an automaton that can verify, for example, that the tiles bcbc, cdbd, bccd, cddc in Figure **??** all meet one another.

Figure 3: Addressing tiles in a supertile

This structure may have practical uses; often one constructs grids to carry out numerical simulations. For various reasons, one may want a grid that is irregular in some fashion. These hierarchical grids require less memory to store (especially give the techniques of Section **??**) as one does not need pointers between adjacent addresses (on the other hand, finding adjacencies each time they are needed obviously carries some computational cost.)

This paper is divided into three main parts:

First in Section **??** we give definitions of substitution tilings, substitution graphs, etc. In Section **??** we define addresses, labelings, etc.

Second, in Section **??** we show infinite-level supertiles actually are closely related to substitution tilings, proving for example that every substitution tiling is the union of infinite level supertiles with disjoint interiors. We give characterizations, in terms of addresses, of familiar properties of substitution species—being non-vacant, having unique decomposition, having connected hierarchy, being repetitive and being strongly repetitive.

Finally, in the following sections we give three interesting applications of addresses:

1. In Section **??** we give a method of producing explicit and detailed descriptions of orbits in any substitution species,

2. In Section **??** we show how to produce automatic descriptions of adjacencies of tiles, supertiles, and infinite-level supertiles, etc.

3. and finally in Section **??** we show how to encode infinite addresses with

locally-finite information, a technique at the heart of the recent production of matching rules enforcing every substitution species [?].

In a future paper or papers, we hope to give further applications:

1. We will derive a close relationship between mutually decomposable species of substitution tilings and automatic translations between their underlying addresses
2. We will show a interesting trick for visualizing certain "palindromic" species of substitution tilings
3. We will develop a series of interesting parallels between geometric groups, subgroups, perfect colorings, group graphs and free groups into the setting of semigroups realized as substitution tilings.
4. Finally, we hope in the future to fully understand the following conjecture:

**Conjecture** Up to some sort of reasonable conditions, the abstract structure given by a set of addresses $\mathbb{A}$ and equivalences $\approx$ completely determines the geometry of a substitution tiling and the ambient space in which it resides.

# 1 The Setting

## 1.1 Substitution Tilings

The general setting is much broader than that taken here. We actually need only a few key assumptions, discussed in more detail in Section **??**. For now these assumptions are subsumed into the current setting.

Let $M = \mathbb{R}^n$, real affine space, endowed with the usual metric and usual measure $\mu$; let $G$ be some group of measure-preserving affine transformations on $M$, with identity $e$. ?

The following term is adapted from geometric topology: this needs work!

**Definition 1.1.1** A point set $A$ in a measurable metric space is **tame** iff $A$ is measurable and closed, has interior with non-zero measure and boundary with zero measure, and all points in the boundary of $A$ are limit points of the interior of $A$.

?

**Definition 1.1.2** Let $T$ be a finite set of **prototiles**— bounded, tame sets in $M$.

**Remarks 1.1.3** The exact conditions we need on the elements of $T$ are somewhat unclear. Certainly smooth or piecewise-flat complexes are perfectly fine, as are tiles with fractal boundary, etc.

We may assume without loss of generality that the elements of $T$ are mutually disjoint in $M$. We do not assume that no pair of elements of $T$ are congruent (this allows many interesting examples and has negative repercussions only in Section **??**). Note that the prototiles have finite measure since compact.

We may wish to "mark" our prototiles by mapping the points in the tiles to some space of markings. A formal description of markings is given in Appendix C of [**?**]; all of the following applies whether markings are used or not, and for now further mention of markings will be suppressed; if markings are used, we will say two tiles are congruent only if there is an element of $G$ taking the points *and* the markings of one tile to the other.

**Definition 1.1.4** **Tiles** will be the images of prototiles under $G$. A **tiling** of any $M' \subseteq M$, is a representation of $M'$ as the union of tiles with disjoint interiors.

**Convention 1.1.5** Both tiles and tilings are sets of (perhaps marked) points in $M$. The symbols $=$, $\subset$, $\cup$, $\cap$ etc. will take on additional meaning when applied to tiles and tilings. For tilings $\tau_1$, $\tau_2$, when we we write, say, $\tau_1 \subset \tau_2$ not only are we implying that the points in $\tau_1$ are among the points in $\tau_2$, but that every tile in $\tau_1$ exactly coincides with a tile in $\tau_2$. Similarly, for a collection of tiles or tilings $\tau_i$, when we write $\cup_i \tau_i$ we imply that this union is also a tiling– no pair of tiles in this union have intersecting interiors.

We thus (awkwardly) denote a tiling of $M'$ as $\cup g_i B_i$, with the understanding that each $g_i \in G$, each $B_i \in T$ and the $g_i B_i$ have disjoint interiors and $M' = \cup g_i B_i$.

**Remark 1.1.6** Note that tiles and tilings are tame and that tiles are bounded and have finite measure since bounded and closed. Note also that we have defined tilings in such a way to include what are commonly called "configurations" as tilings of subsets of $M$.

**Definition 1.1.7** Next, let $\sigma$ be an **inflation**, any distance increasing affine transformation from $M$ to $M$, such that $\sigma$ has a unique fixed point $\circ$— the **origin**— and the cyclic group generated by $\sigma$ is normal in $< \sigma, G >$, the subgroup of all affine transformations from $M$ to $M$ generated by elements of $G$ and $\sigma$.

In particular, for all $g \in G$, for any integer $n$, there exists $h \in G$ with $\sigma^n g = h\sigma^n$. [2]

We define for each $g \in G$, a family $\{g^{(n)} \mid n \in \mathbb{Z}\} \subset G$ such that $g^{(0)} = g$ and $\sigma^n g = g^{(n)}\sigma^n$. It soon follows that, for all $n, m \in \mathbb{Z}$:
$\sigma^n g^{(m)} = g^{(n+m)}\sigma^n$
$(g^{(m)})^{(n)} = g^{(n+m)} = (g^{(n)})^{(m)}$ and
$(g^m)^{(n)} = (g^{(n)})^m$

**Definition 1.1.8** A **substitution** is an automorphism $\sigma'$ on the set of tilings of subsets of $M$ by prototiles $T$ under congruences $G$, such that for any tiling $\tau$ of $M'$, $g \in G$,
(i) $\sigma'(\tau)$ is a tiling of $\sigma(M')$.
(ii) $\sigma'(g\tau) = g^{(1)}\sigma'(\tau)$

Note by (ii) above, since $\sigma$ defined on the prototiles themselves, that for tilings $\tau_1$, $\tau_2$ such that $\tau_1 \cup \tau_2$ is a tiling of some subset of $M$, we have $\sigma'(\tau_1) \cup \sigma'(\tau_2) = \sigma'(\tau_1 \cup \tau_2)$.

For each $A \in T$, define appropriate $\{g_{Ai}\} \subset G$ and $\{B_{Ai}\} \subset T$ in order to set

$$\sigma'(A) := \cup g_{Ai} B_{Ai} \tag{1}$$

(This notation is improved in equation **??** below)

**Remark 1.1.9** Without loss of generality, we assume the elements of $T$ are disjoint from each tile in each $\sigma'(A)$. (This makes certain things nicer in defining

[2] This follows from normality as $g, h$ are measure preserving and $\sigma$ is not; for more general $G$, we must explicitly give this as a condition on $\sigma$

$S$ below). Note that

$$\mu(w(A) = \mu(w'(A)) = \sum g_{Ai} B_{Ai} = \sum B_{Ai}$$

Note we do not require every element of $T$ to appear in some $\sigma'(B)$, $B \in T$ as there are many useful examples that do not satisfy this seemingly minimal requirement (See Theorem **??**).

For any tile $gA$, $g \in G$, $A \in T$, we have, as a tiling of $g^{(1)}\sigma(A) = \sigma(gA)$:

$$\sigma'(gA) = g^{(1)}\sigma'(A) = g^{(1)}(\cup g_{Ai} B_{Ai})$$

**Definition 1.1.10** Given $T$ and $w'$, for any $A \in T$, $g \in G$, $n \in \mathbb{N}$ we inductively define $n$**-level supertiles** $(\sigma')^n(A)$,$(\sigma')^n(gA)$ as tilings of $\sigma^n(A)$, $\sigma^n(gA)$ resp (see also equation **??** below):

$$(\sigma')^n(A) := (\sigma')^{n-1}(\cup(g_{Ai} B_{Ai})) = \cup g_{Ai}^{(n-1)}(\sigma')^{n-1}(B_{Ai}) \qquad (2)$$

Thus,

$$(\sigma')^n(gA) = g^{(n)}(\sigma')^{n-1}(\cup(g_{Ai} B_{Ai})) = g^{(n)}(\cup g_{Ai}^{(n-1)}(\sigma')^{n-1}(B_{Ai}))$$

And now the the most important definition of all:

**Definition 1.1.11** The **substitution species** $\Sigma(T, \sigma')$ of substitution tilings arising from $T$, $\sigma'$ is the collection of tilings $\tau$ of $M$ by images of elements of $T$ such that any bounded tiling in $\tau$ is congruent to tiling in some supertile $(\sigma')^n(A)$.

As discussed in our introduction, this certainly gives a nice local description of a tiling in the species, but at this point, even to show the species is nonempty requires work. We rectify this below, and ultimately give a complete *global* description of the species. For example it is quite natural to wonder about the orbits in $\Sigma(T, \sigma')$ under $\sigma'$; these are self-evident with addressing.

**Example 1.1.12** We have one example in Figure **??**; here is another (Figure **??**). $T$ consists of the three triangles shown $\sigma$ could be regarded as some dilation of magnitude $s \approx 1.324717957244746$, the real root of $s^3 - s - 1 = 0$. $\sigma'$ takes the elements of $T$ to supertiles as shown. A small portion of a substitution tiling in $\Sigma(T, \sigma')$ is at right.

Figure 4: A substitution on triangles

We pause for further useful definitions:

**Definition 1.1.13** A tiling $\tau$ is **periodic** iff there exists $g \in G$ such that $g\tau = \tau$. A tiling is **non-periodic** iff it is not periodic. A species is **aperiodic** iff every element of the species is non-periodic.

Aperiodic (species of) tilings have of course received great attention; eg. [**?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**] etc., but are outside our focus at the moment.

**Definition 1.1.14** A substitution species $\Sigma(T, \sigma')$ has **unique decomposition** iff for all $\tau' \in \Sigma(T, \sigma')$, there exists unique $\tau$ such that $\sigma'(\tau) = \tau'$.

That is, $\Sigma(T, \sigma')$ has unique decomposition if and only if $\sigma'$ is one-to-one on $\Sigma(T, \sigma')$.

In Theorem **??** we explicitly prove that $\sigma' : \Sigma(T, \sigma') \to \Sigma(T, \sigma')$ is onto.

It is easy to show that if there is a periodic tiling in $\Sigma(T, \sigma')$ then $\Sigma(T, \sigma')$ does *not* have unique decomposition. Solomyak [**?**] showed that if $G$ consists of translations only, then if $\Sigma(T, \sigma')$ is aperiodic then $\Sigma(T, \sigma')$ has unique decomposition. This author provided an example of an aperiodic $\Sigma(T, \sigma')$ that does not have unique decomposition [**?**]. The general situation remains unclear.

**Definition 1.1.15** A tiling $\tau \in \Sigma(T, \sigma')$ has **connected hierarchy** iff for all tiles $g_1 A_1, g_2 A_2 \in \tau$, there exists a supertile $g(\sigma')^n(A)$ in $\tau$ containing both $g_1 A_1, g_2 A_2$. If a tiling does not have connected hierarchy it is useful to think of "an infinite fault line" running through the tiling.

However, until we define addresses it is a hassle to prove each $\Sigma(T, \sigma')$ has both tilings with and without connected hierarchy, that infinite fault lines really exist, as boundaries between infinitely large "supertiles", much less to exploit this to give a simple indexing of the elements of $\Sigma(T, \sigma')$ itself as we do below. Incidentally, we do note that the subset of $\Sigma(T, \sigma')$ having connected hierarchy has measure 1 in any translation invariant probability measure of $\Sigma(T, \sigma)$ ([**?**], Appendix C of [**?**]).

**Definitions 1.1.16** A *tiling* tiling $\tau$ of $M$ is **repetitive** if and only if for every bounded tiling $\tau'$ within $\tau$ there exists an $R \in \mathbb{R}$, $R > 0$ such that every ball of radius $R$ contains an image in $G$ of $\tau'$ in $\tau$.

A *tiling* $\tau$ of $M$ is **strongly repetitive** ask Marjorie

if and only if for every $r \in \mathbb{R}$, $r > 0$ there exists an $R \in \mathbb{R}$, $R > 0$ such that for every tiling $\tau'$, of diameter less than $r$, within $\tau$, every ball of radius $R$ in $M$ contains an image in $G$ of $\tau'$ in $\tau$.

A *species* $\Sigma$ of tilings of $M$ is **repetitive** if and only if for every bounded tiling $\tau'$ within some $\tau \in \Sigma$ there exists an $R \in \mathbb{R}$, $R > 0$ such that for every $\tau'' \in \Sigma$ every ball of radius $R$ contains an image in $G$ of $\tau'$ in $\tau''$.

A *species* $\Sigma$ of tilings of $M$ is **strongly repetitive** if and only if for every $r \in \mathbb{R}$, $r > 0$ there exists an $R \in \mathbb{R}$, $R > 0$ such that for every bounded tiling $\tau'$, of diameter less than $r$, within some $\tau \in \Sigma$, for every $\tau'' \in \Sigma$, every ball of radius $R$ contains an image in $G$ of $\tau'$ in $\tau''$.

We give simple, exact conditions under which each of these properties holds in Theorem **??**.

## 1.2 The Substitution Graph

**Definition 1.2.1** It is useful to draw a **substitution graph** $\Gamma(T, \sigma')$, a directed graph with nodes indexed by $T$ : Each prototile is represented by a node of the graph; if for $A \in T$, $\sigma'(A) = \cup(g_{Ai} B_{Ai})$, then directed edges (arrows) departing the node $A$ are indexed by the tiles $\{g_{Ai} B_{Ai}\}$ and head towards the nodes indexed $\{B_{Ai}\}$.

We call the set of tiles indexing the labeled arrows $S$ (Note the elements of $S$ are in 1-1 correspondence with the arrows of $\Gamma(T, \sigma')$ by Remark **??**). There is a natural projection from $S$ to $T$: for any $A \in S$, let $A_T \in T$ be the element of $T$ identified with the node at the head of $S$.

The following conventions are enormously useful. They should be interpreted by viewing the elements of $S$ as elements of $T$ who "know their parents" (i.e. identify each $A \in S$ with $A_T \in T$; but keep in mind that an element of $S$ has a particular position with respect to some 1-level supertile– its "parent".)

For $A \in T$, the collection of nodes at the tail end of arrows with head at node $A$ is denoted $A^- \subset T$.

For $A \in S$, the single node at the tail end of arrow $A$ is denoted $A^- \in T$.

For $A \in S$, the single node at the head of arrow $A$ is denoted $A_T \in T$;

Thus, for $A \in T$, $A^- = \{B^- \mid B \in S, B_T = A\}$.

For $A \in T$, the collection of arrows departing node $A$ is denoted $A^+ \subset S$

For $A \in S$ the collection of arrows departing node $A_T$ is denoted $A^+ \subset S$.

That is, for $A \in S$, $A^+ = (A_T)^+$.

Finally, for each $B \in S$, $B = g_{Ai} B_{Ai}$ for some $i$ and $A = B^-$. Let $g_B = g_{Ai}$, and note that $B_{Ai} = B_T$

For later convenience, for $A \in T$, let $g_A = e \in G$ and $A_T = A$.

Our conventions allow us to restate equations **??** and **??** in much cleaner form. We now have, for $A \in T$, $n \geq 1$, $g \in G$:

$$\sigma'(A) \quad := \quad \bigcup_{B \in A^+} B \quad = \quad \bigcup_{B \in A^+} g_B B_T \tag{3}$$

and more generally,

$$(\sigma')^n(gA) \quad := \quad g^{(n)} \bigcup_{B \in A^+} g_B^{(n-1)} (\sigma')^{n-1}(B_T) \tag{4}$$

**Example 1.2.2** We continue Example **??**. At left on Figure **??**, we have labeled our prototiles $T = \{X, Y, Z\}$ and $S = \{A, B, C, D, E\}$. At right the substitution graph $\Gamma(T, \sigma')$ is shown. We have not really indicated the isometries $g_A, g_B$, etc.

Note that we have, for example: $Z = Z_T = D_T = C_T$ $Z^+ = \{E, D\} = D^+ = C^+$ $Z^- = \{Y, Z\}$ but $C^- = Y$ and $D^- = Z$.

Figure 5: A substitution graph

We make the following elementary but important observations:

**Lemma 1.2.3** *Given $T$, $\sigma'$:*

*(i) $\Gamma(T, \sigma')$ has a finite number of nodes, and from each node at least one arrow departs.*

*(ii) $S$ is finite.*

*(iii) $|S| > |T|$*

**Proof** (i) This follows trivially from the identification of the nodes of $\Gamma(T, \sigma')$ with the elements of $T$ and that $\sigma'$ is defined for each element of $T$.

(ii) Because each element $A$ of $T$ has defined, finite positive measure, and $\sigma$ and the elements of $G$ are affine, $\sigma(A)$ and the elements of $A^+$ also have finite positive measure. Hence there can be a most finitely many $S$.

(iii) By Remark **??** the elements of $\{\sigma(A) \mid A \in T\}$ are disjoint; hence $|S| \geq |T|$. Now $\sigma$ increases all distances and so increases measure. There exists $A \in T$ such that $\mu(A) \geq \mu(B)$ for all $B \in T$. $\mu(\sigma(A)) = \sum_{B \in A^+} \mu(B) > \mu(A')$ for all $A' \in T$; however, for each $B \in A^+$, $\mu(B) = \mu(A')$ for some $A' \in T$; hence $|A^+| > 1$ and $|S| > |T|$. □

# 2 Addresses

We now come to our central definition.

**Definition 2.0.4** An **address** is a map $\mathcal{A} : \mathbb{Z} \to T \cup S \cup \{\oslash\}$,

(i) $\oslash$ is a symbol having no previous connection to $T$ or $S$.

(ii) There exists at most one $k \in \mathbb{Z}$ with $\mathcal{A}(k) \in T$ and at least one $k \in \mathbb{Z}$ with $\mathcal{A}(k) \in T \cup S$.

(iii) For any $k \in \mathbb{Z}$, if $\mathcal{A}(k) \in \{\oslash\} \cup T$, $\mathcal{A}(k+1)$ must equal $\oslash$.

(iv) For any $k \in \mathbb{Z}$, if $\mathcal{A}(k) \in T \cup S$, $\mathcal{A}(k-1)$ must be an element of $\mathcal{A}(k)^+ \subset S$

Let $\mathbb{A}$ be the set of all addresses admitted by $\Gamma(T, \sigma')$.

**Remarks 2.0.5**

For any $k \in \mathbb{Z}$, if $\mathcal{A}(k) \in S$, then $\mathcal{A}(k+1) = \oslash$, or $\mathcal{A}(k+1) = \mathcal{A}(k)^- \in T$, or $\mathcal{A}(k+1) \in S$ with $\mathcal{A}(k+1)_T = \mathcal{A}(k)^- \in T$. If $\mathcal{A}(k) \in T$, then $\mathcal{A}(k+1) \in \oslash$.

It is *quite* handy to have a good notational shorthand:

We might consider addresses as bi-infinite strings of digits in $T$, $S$, perhaps with a leading infinite string of the special digit $\oslash$.
To fix these strings relative to the integers, we write $A_n$ for $\mathcal{A}(n) \in S \cup T$. The subscript will *always* imply position.

Thus a typical string might be denoted $\mathcal{A} = A_n...$ We will generally drop the symbol $\oslash$; in an address represented by $...A_k...$ it is implicit that $\oslash$ does not appear.

It is also useful to denote *sets* of addresses, and introduce a second special character $\star$, representing all possible infinite strings on the right, eg:

$$A_n\star = \{\mathcal{A} \in \mathbb{A} \mid \mathcal{A}(n) = A_n, \mathcal{A}(k > n) = \oslash\}$$

and

$$...A_0\star = \{\mathcal{A} \mid \mathcal{A}(n) = A_n \ \forall \ n \geq 0\}$$

$$A\star = \{\mathcal{A} \in \mathbb{A} \mid \exists n \ \text{ s.t. for } \ A_n = A, \mathcal{A} = A_n\star\}$$

We will loosely refer to strings of this form as addresses as well.

For each $n \in \mathbb{Z}$ let

$$\mathbb{A}^n = \{A_n... \in \mathbb{A}\} = \{\mathcal{A} \in \mathbb{A} \mid \mathcal{A}(n) \neq \oslash = \mathcal{A}(n+1)\}$$

$$\mathbb{A}^\infty = \{...A_k...\} = \mathbb{A} - (\cup_n \mathbb{A}^n)$$

.

We can use subscripts to denote sets of addresses described by $\star$; for example:

$$\mathbb{A}^n_m = \{A_n...A_m\star \subset \mathbb{A}^n\}$$

$$\mathbb{A}_m = \cup_n \mathbb{A}^n_m \cup \{...A_m\star\}$$

When working with a *specific* example, as in Figure **??**, Examples **??** and **??**, subscripts become awkward. Suppose $T = \{X, Y\}$, $S = \{B, C, D\}$. The string $\mathcal{A} = A_2...A_{-1}\star$ does not give specific values for each $A_n$; but indicating these values by, say, $\mathcal{A} = C_2C_1D_0B_{-1}\star$ seems rather odd. In this case, rather than using subscripts as above, we make use of a **decimal point $\bullet$**, placed immediately to the right of $\mathcal{A}(0)$, and write out the specific digits of $\mathcal{A}$. Thus, for example,

$$CCD\bullet B\star := \{A_2...A_{-1}... \mid A_2 = A_1 = C, A_0 = D, A_{-1} = B\} \in \mathbb{A}^2_{-1}$$

We can also use the symbol $\star$ to fix the position of digits a few places to the left of $\bullet$:

$$C{\star}{\bullet}{\star} := \{A_1 \ldots \mid A_1 = C\} \in \mathbb{A}_1^1$$

Finally, for addresses built from repeating strings, we use a bar as when representing expansions of rational numbers:

$$C{\bullet}D\overline{AB} = C{\bullet}DABABABAB\ldots$$

$$\overline{ABC}D{\bullet}C{\star} = \ldots ABCABCABCD{\bullet}C{\star}$$

**Definition 2.0.6** We define a **shift** $\varsigma : \mathbb{A} \to \mathbb{A}$ by $(\varsigma(\mathcal{A}))(n) = \mathcal{A}(n-1)$.

That is, $\varsigma$ literally shifts addresses (represented as strings as above) to the left. Note, of course, $\varsigma$ is 1-1 and onto. As it should be, this operation is closely linked to $\sigma$, as seen in equations **??**, **??**, **??** and **??** below.

## 2.1 Labeling points and tiles

Addresses are automatic– a given set $\mathbb{A}$ can essentially be regarded as a regular language, a language described by an automaton. Indeed, the needed automaton is closely related to the substitution graph $\Gamma(T, \sigma')$ itself. This is further discussed in Section **??**.

In the meantime, we describe how addresses can be used, first to identify points in tiles and supertiles through (see Equation **??**)

$$\lambda : \cup_{n \in \mathbb{Z}} \ \ A_{-\infty}^{\ n} \to M$$

In Section **??**, we go on to identify tiles and supertiles in $M$ through (see Equation **??**)

$$\lambda' : \cup_{n,m \in \{0,1\ldots\}} \ \ A_m^n \to \{g(\sigma')^k(A) \mid g \in G, k \in \{0,1,\ldots\}, A \in T\}$$

and through, for each $h \in G$ (see Equation **??**):

$$\lambda_h' : \cup_{n,m \in \{0,1\ldots\}} \ \ A_m^n \to \{g(\sigma')^k(A) \mid g \in G, k \in \{0,1,\ldots\}, A \in T\}$$

and finally to construct, explicitly, infinite-level supertiles

$$\lambda_h'(\ldots A_m \star)$$

which we will see more or less correspond to tilings in $\Sigma(T, \sigma')$.

We also describe two equivalence relations $\sim$ and $\approx$ and a relation $|$ which play an important role in giving the precise correspondence between infinite-level supertiles and $\Sigma(T, \sigma')$.

An interesting point to consider is that without these relations, $\mathcal{A}$ is essentially structureless. The degree to which these relations *force* the geometry of $\Sigma(T, \sigma')$ is an open and compelling question.

**Definition 2.1.1** We first define, for any $A_n....A_m\star \in \mathbb{A}_m^n$, the highly useful

$$g_{A_n...A_m\star} := g_{A_n}^{(n)}...g_{A_m}^{(m)} \tag{5}$$

For insight into the meaning of this device, see Lemma **??**, or Figure **??**. If $n = 0$, $g_{A_0\star} = g_{A_0}$; thus when $n = 0$ or $n \neq m$ there is no ambiguity in dropping the symbol $\star$ from subscripts of elements of $G$ (however, in general $g_{A_n\star} = g_{A_n}^{(n)} \neq g_{A_n}$). Note too that

$$g_{A_n...A_{k+1}A_k}\sigma^k((A_k)_T) = g_{A_n...A_{k+1}}\sigma^k(A_k) \tag{6}$$

**Lemma 2.1.2** *For $A_n... \in \mathbb{A}^n$, for all $k \leq n$,*

$$g_{A_n...A_k}\ \ \sigma^k(\mathcal{A}(k)_T) \supset g_{A_n....A_kA_{k-1}}\ \ \sigma^{k-1}(\mathcal{A}(k-1)_T) \tag{7}$$

**Proof** For any $k \leq n$, $\mathcal{A}(k-1) \in \mathcal{A}(k)^+$; thus by equation **??**

$$\sigma(\mathcal{A}(k)_T) \supset g_{\mathcal{A}(k-1)}(\mathcal{A}(k-1))_T$$

and for any $h \in G$,

$$h\ \sigma^k(\mathcal{A}(k)) \supset h\ g_{\mathcal{A}(k-1)}^{(k-1)}\ \sigma^{k-1}(\mathcal{A}(k-1))_T$$

In particular

$$g_{A_n...A_k}\ \ \sigma^k(\mathcal{A}(k)_T) \supset g_{A_n....A_kA_{k-1}}\ \ \sigma^{k-1}(\mathcal{A}(k-1)_T)$$

□

**Definition 2.1.3** We now define for $\mathcal{A} = A_n.... \in \mathbb{A}^n$:

$$\lambda(\mathcal{A}) := \bigcap_{k<n} g_{A_n...A_{k+1}}\sigma^k(A_k) = \bigcap_{k\leq n} g_{A_n...A_k}\sigma^k((A_k)_T) \tag{8}$$

**Lemma 2.1.4** *For $n \in \mathbb{Z}$, $\mathcal{A} = A_n.... \in \mathbb{A}^n$, $\lambda(\mathcal{A})$ is a well-defined unique point in $M$. Moreover,*

$$\lambda(\varsigma(\mathcal{A})) = \sigma(\lambda(\mathcal{A})) \tag{9}$$

**Proof** By Lemma **??** the $g_{A_n...A_{k-1}} \ \sigma^k(A_k)$ are nested closed sets; since $M$ is a complete space, their intersection is non-empty.

Now note that since $\sigma^{-1}$ *reduces* all distances and so as $k \to -\infty$, $diam(\sigma^k(\mathcal{A}(k)) \to 0$, as do the diameters of any images of $\sigma^k(\mathcal{A}(k))$. Since $M$ is metric, $M$ is Hausdorff ($T_2$), and there is only a single point in $\lambda(\mathcal{A})$.

As to the proof of equation **??**, let $B_{n+1}... = \varsigma(\mathcal{A})$; thus for all $k \leq n$, $B_{k+1} = A_k$. Thus

$$\begin{aligned} \lambda(\varsigma(\mathcal{A})) &= \bigcap_{k \leq (n+1)} g_{B_{n+1}...B_k} \sigma^k((B_k)_T) \\ &= \bigcap_{k \leq n} (g_{A_n...A_k})^{(1)} \sigma^{k+1}((A_k)_T) \\ &= \sigma(\bigcap_{k \leq n} (g_{A_n...A_{k+1}}) \sigma^k((A_k)_T) \\ &= \sigma(\lambda(\mathcal{A})) \end{aligned}$$

□

**Lemma 2.1.5** *For any $n \in \mathbb{Z}$, $A \in S \cup T$ $g \in G$,*

$$\lambda(A_n\star) = \sigma^n(A) \tag{10}$$

*More generally for any $n, m \in \mathbb{Z}$, $n \geq m$, and any $A_n...A_m\star \in \mathbb{A}^n_m$*

$$\lambda(A_n...A_m\star) = g_{A_n...A_{m-1}} \sigma^m(A_m) \tag{11}$$

The proof is a trivial exercise in manipulating the notation. See archive/addresses.2.17.97

**Lemma 2.1.6** *For any $A \in S \cup T$, $n \in \mathbb{N}$, the supertile $(\sigma')^n(A)$ is exactly tiled by the sets $g_{A_n...A_1} A_0 = \lambda(A_n...A_0\star)$ with $A_n = A$, $A_n...A_0\star \in \ A_n\star$*

That is, the tiles in the tiling $(\sigma')^n(A)$ correspond exactly with the sets $\lambda(A_n...A_0\star)$.

**Proof** Let $A_n = A$. Then

$$(\sigma')^n(A) \quad = \quad g^{(n)}_A (\sigma')^n(A_T)$$

by (**??**)

$$
\begin{aligned}
&= g_A^{(n)}(\sigma')^{n-1}(\bigcup_{B\in A^+} g_B B_T) \\
&= \bigcup_{\substack{A_nA_{n-1}\star \\ \subset A_n\star}} g_{A_nA_{n-1}} (\sigma')^{n-1}((A_{n-1})_T) \\
&= \bigcup_{\substack{A_nA_{n-1}A_{n-2}\star \\ \subset A_n\star}} g_{A_nA_{n-1}A_{n-2}} (\sigma')^{n-2}((A_{n-2})_T) \\
&= \ldots \\
&= \bigcup_{\substack{A_n\ldots A_0\star \\ \subset A_n\star}} g_{A_n\ldots A_1} A_0 \qquad (12)
\end{aligned}
$$

by (**??**)

$$
= \bigcup_{\substack{A_n\ldots A_0 \\ \star\subset A_n\star}} \lambda(A_n\ldots A_0\star) \qquad (13)
$$

Note that at every stage, the union is of affine images with disjoint interiors of elements of $T$. In the final two lines, these images are tiles; thus these last unions are tilings. □

**Definitions 2.1.7** Define an equivalence relation $\approx$ on addresses in $\cup_{n\in\mathbb{Z}}\ \mathbb{A}^n_{-\infty}$ as follows:

$\mathcal{A}\approx\mathcal{B}$ if and only if
(i) $\exists n\in\mathbb{Z}$ such that $\mathcal{A},\mathcal{B}\in\mathcal{A}^n_{-\infty}$ and $\mathcal{A}(n)=\mathcal{B}(n)$
(ii) $\lambda(\mathcal{A})=\lambda(\mathcal{B})$

That is, $\mathcal{A}\approx\mathcal{B}$ if and only if $\mathcal{A}$ and $\mathcal{B}$ describe the same point in some $\sigma^n(A)$, $A=\mathcal{A}(n)=\mathcal{B}(n)$.

We next define a symmetric, reflexive but not transitive relation $\mid$ on addresses in $\cup_{n,m\in\mathbb{Z}}\ \mathbb{A}^n_m$:

For $\mathcal{A},\mathcal{B}\in\ \cup_{n,m\in\mathbb{Z}}\ \mathbb{A}^n_m$, $\ \mathcal{A}\mid\mathcal{B}$ if and only if

(i) $\exists n, m \in \mathbb{Z}$ such that $\mathcal{A}, \mathcal{B} \in \mathcal{A}_m^n$ and $\mathcal{A}(n) = \mathcal{B}(n)$
(ii) $\lambda(\mathcal{A}) \cap \lambda(\mathcal{B}) \neq \emptyset$ but $\lambda(\mathcal{A}) \neq \lambda(\mathcal{B})$

That is, $\mathcal{A} \mid \mathcal{B}$ if and only if $\mathcal{A}$ and $\mathcal{B}$ describe adjacent tiles in $\sigma^n(A)$, $A = \mathcal{A}(n) = \mathcal{B}(n)$. Note that $\mathcal{A} = A_n \ldots A_m \star \mid A'_n \ldots A'_m \star = \mathcal{B}$ if and only if there exist $A_n \ldots A_m \ldots \in \mathcal{A}$, $A'_n \ldots A'_m \ldots \in \mathcal{B}$ with $A_n \ldots \approx A'_n \ldots$.

Here $\mid$ is defined on certain addresses; Definition **??** defines $\mid$ on certain supertiles and tilings.

**Example 2.1.8** An example may bring all this notation to life.

We saw the $L$-tiles in figures **??**, **??** and **??**. Consider now Figure **??**

The origin $\mathsf{o}$ is marked with an asterisk; the inflation $\sigma$ doubles all coordinates. $T$ consists of one tile, the $L$-tile, denoted $\mathsf{x}$, in the position indicated. The four tiles in $S$ are denoted $\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}$, in the same positions relative to $\sigma(x)$ as in Figure **??**. (Note that we take our group $G$ to be isometries generated by rotations and translations) And so, we have the indicated structures. Keep in mind the conventions described in Remark **??**.

Figure 6: Addressing points and sets of points

## 2.2 Addressing tilings; constructing infinite level-supertiles

Just as we had the maps $\sigma$ between points in $M$ and $\sigma'$ between tilings in $M$, we use $\lambda$, a map from addresses to points, to define a map $\lambda'$ between certain sets of addresses and tilings:

**Definition 2.2.1** For $n, m \in \mathbb{Z}$, $n \geq m \geq 0$ define

$$\lambda'(A_n \ldots A_m \star) \; := \; g_{A_n \ldots A_{m+1}} (\sigma')^m (A_m) \tag{14}$$

$$= g_{A_n...A_m}(\sigma')^m((A_m)_T) = \bigcup_{\substack{A_n...A_0\star \\ \subset A_n...A_m\star}} \lambda(A_n...A_0\star)$$

That is, $\lambda'(A_n...A_m\star)$ precisely corresponds to an image of supertile $(\sigma')^m((A_m)_T)$; this image is in "position" $A_n...A_m$ in the supertile $(\sigma')^n(A_n)$. In particular,

$$\lambda'(A_n\star) = (\sigma')^n(A_n) \tag{15}$$

Note $\lambda'(A_0\star) = A_0$.

We thus have

**Lemma 2.2.2** *For $n, m \in \mathbb{N}$, $m \leq n$,*

$$\lambda'(\varsigma(A_n...A_m\star) = \sigma'(\lambda'(A_n...A_m\star) \tag{16}$$

The proof is a routine manipulation of notation.

**Definitions 2.2.3** For $h \in G$, $A_n...A_m\star \in \mathbb{A}^n_m$,$n \geq 0$ we define:

$$\begin{aligned} \lambda'_h(A_n...A_m\star) &:= h(g_{A_n...A_{m+1}})^{-1}\lambda'(A_n\star) \\ &= h(g_{A_n...A_{m+1}})^{-1}(\sigma')^n(A_n) \end{aligned} \tag{17}$$

(If $n = m$ simply use $\lambda'_h(A_n\star) := \lambda'(A_n) = (\sigma')^n(A_n)$)
Note that we have essentially shifted the tiling $\lambda'(A_n\star)$ so that the supertile $\lambda(A_n...A_m\star)$ is sent to $h(\sigma')^m(A_m)$:

$$\begin{aligned} \lambda'_h(A_n...A_m\star) &= h(g_{A_n...A_{m+1}})^{-1}\lambda'(A_n\star) \\ &\subset h(g_{A_n...A_{m+1}})^{-1} \; g_{A_n...A_{m+1}}(\sigma')^m(A_m) \\ &= h(\sigma')^m(A_m) \end{aligned} \tag{18}$$

Note

$$\lambda'_e(A_n...A_m\star) \neq \lambda'(A_n...A_m\star)$$

For one thing, the left-hand side is congruent to $(\sigma')^n(A_n)$; the right-hand side is congruent to $(\sigma')^m(A_m)$.

We now come to a very important definition: For $h \in G$, $m \in \mathbb{Z}$, $...A_m\star \in \mathbb{A}^\infty_m$, define:

$$\lambda'_h(...A_m\star) := \bigcup_{k \geq m} \lambda'_h(A_k...A_m\star) \tag{19}$$

Finally define the equivalence relation $\sim$ on $\{\lambda'_h(...A_m\star) \mid h \in G, m \in \mathbb{Z}, \ldots A_{m\star} \in A^{\infty}_m\}$ by, for $h, h' \in G$, $...A_m\star, ...A'_{m'}\star \subset \mathbb{A}^{\infty}$,

$\lambda'_h(...A_m\star) \sim \lambda'_{h'}(...A'_{m'}\star)$

if and only if there exists an $n \in \mathbb{N}$, $n > m, m'$ such that for all $k > n$, $A'_k = A_k$ and such that $h(g_{A_n \ldots A_m})^{-1} = h'(g_{A'_n \ldots A'_{m'}})^{-1}$

Equivalence classes under $\sim$ of the set of $\lambda'_h(...A_m\star)$ are called **infinite-level supertiles**. An infinite-level supertile $\mathcal{S}$ implicitly can be described by a representative $\lambda'_h(...A_m\star)$. For infinite-level supertiles $\mathcal{S}, \mathcal{S}'$, when we write $\mathcal{S} = \mathcal{S}'$, they are equivalent as tilings (see the following Lemma); when we write $\mathcal{S} \sim \mathcal{S}'$ they are equivalent as hierarchies of tilings (i.e., $\mathcal{S}, \mathcal{S}'$ can be represented by $\lambda'_h(...A_m\star), \lambda'_{h'}(...A'_{m'}\star)$ with $\lambda'_h(...A_m\star) \sim \lambda'_{h'}(...A'_{m'}\star)$)

Thus, by the following Lemma, $\mathcal{S} \sim \mathcal{S}'$ implies $\mathcal{S} = \mathcal{S}'$ but the converse only holds when $\Sigma(T, \sigma')$ has unique decomposition (See Theorem **??**).

**Lemma 2.2.4** *For $h \in G$, $n, m \in N$, $n > m$, $A_n....A_m\star \in \mathbb{A}^n_m$, as tilings we have:*

$$\lambda'_h(A_n...A_m\star) \supset \lambda'_h(A_{n-1}...A_m\star) \tag{20}$$

*Consequently, for $...A_m\star \in \mathbb{A}^{\infty}_m$, $\lambda'_h(...A_m\star)$ is a well-defined tiling in $M$.*

*Moreover, for $h, h' \in G$, $...A_m\star, ...A'_{m'}\star \subset \mathbb{A}^{\infty}$, if $\lambda'_h(...A_m\star) \sim \lambda'_{h'}(...A'_{m'}\star)$ then $\lambda'_h(...A_m\star) = \lambda'_{h'}(...A'_{m'}\star)$ as tilings.*

The proof is a routine manipulation of notation.

**Remark 2.2.5** In particular, **note the total lack of any diagonalization argument, and the explicit nature of the construction of infinite-level supertiles.**

**Lemma 2.2.6** *Let $m \in \mathbb{Z}$, let $...A_m\star \in \mathbb{A}^{\infty}_m$, $h \in G$, and let $h'(\sigma')^m A$, $A \in T \cup S, h' \in G, k \geq 0$ be any supertile in $\lambda'_h(...A_m\star)$. Then there exists an address $...A'_m\star \in \mathbb{A}^{\infty}_m$ such that there exists a $g \in G$ with $gA'_m = A$, and $\lambda'_h(...A_m\star) = \lambda'_{h'g}(...A'_m\star)$.*

The proof is a routine manipulation of notation.

**Lemma 2.2.7** *For any $k, m, n \in \mathbb{Z}$, $n > 0$, $k \geq -n$ for any $\mathcal{A} \in \mathbb{A}^n_m$, $h \in G$*

$$(\sigma')^k(\lambda'_h(A_n...A_m\star) = \lambda'_{h^{(k)}}(\varsigma^k A_n...A_m) \tag{21}$$

*Consequently, we have, for any $k \geq 0$, for any $\mathcal{A} \in \mathbb{A}^{\infty}_m$, $h \in G$*

$$(\sigma')^k(\lambda'_h(...A_m\star)) = \lambda'_{h^{(k)}}(\varsigma^k(...A_m\star)) \tag{22}$$

**Proof**
We begin with $k, m, n \in \mathbb{Z}$, $n > 0$, $k \geq -n$, $\mathcal{A} \in \mathbb{A}^n_m$, $h \in G$.

$$\begin{aligned}
(\sigma')^k \lambda'_h(A_n...A_m\star) &= h^{(k)}(\sigma')^k(g_{A_n...A_{m+1}})^{-1}(\sigma')^n(A_n) \\
&= h^{(k)}((g^{(m+1)}_{A_m})^{-1})^{(k)}...((g^{(n)}_{A_n})^{-1})^{(k)} \quad (\sigma')^{n+k}(A_n) \\
&= h^{(k)}(g^{(m+1+k)}_{A_{m+1}})^{-1}...(g^{(n+k)}_{A_n})^{-1} \quad (\sigma')^{n+k}(A_n) \\
&= h^{(k)}g_{\varsigma^k A_n...A_{m+1}} \quad (\sigma')^{n+k}(A_n) \\
&= \lambda_{h^{(k)}}(\varsigma^k A_n...A_m\star)
\end{aligned}$$

Equation **??** follows immediately from equations **??** and **??**. □

We thus have:

**Theorem 2.2.8** *For every $\lambda'_h(...A_m\star)$ there exists a $\lambda'_{h'}(...A'_{m'}\star)$ with $\sigma'(\lambda'_{h'}(...A'_{m'}\star)) = \lambda'_h(...A_m\star)$.*

*Moreover, if we have $\lambda_{h_1}(\mathcal{A}^1) \sim \lambda_{h_2}(\mathcal{A}^2)$ then their preimages under $\sigma'$ also are equivalent under $\sim$. Thus we can define without ambiguity, for every infinite-level supertile $\mathcal{S}$, an infinite level supertile $\mathcal{S}'$ with $\sigma'(\mathcal{S}') \sim \mathcal{S}$*

**Proof** Simply let $h' = h^{(-1)}$, let $m' = m - 1$ and let $...A'_{m'}\star = \varsigma^{-1}(...A_m)$.

The second observation follows from two simple facts: If $\mathcal{A}_1$ and $\mathcal{A}_2$ agree to the left of some position $n$, so do $\varsigma(\mathcal{A}^1)$ and $\varsigma(\mathcal{A}^2)$; moreover, if $g = h \in G$, $g^{(-1)} = h^{-1}$. □

**Definitions 2.2.9** Recall Definitions **??**. We now define $|$ on $\{\lambda'_h(A_n \ldots A_m\star) \mid h \in G, A_n \ldots A_m\star \subset \mathbb{A}, A_n \in S\}$ by[3]

$\lambda'_h(A_n \ldots A_m\star) \mid \lambda'_{h'}(A'_{n'} \ldots A'_{m'}\star)$ if and only if:
(i) $n = n'$, $A'_{n'} \in \{A_{n'}, (A_{n'})_T\}$

[3]Note the leading digit is taken to be in $S$.

(ii) $\exists\ j > n$, $A_j\star \in \mathbb{A}^j$, $A_j \dots A_n\star, A_j \dots A'_n\star \in A_j\star$ such that
(a) $\lambda'(A_j \dots A_{n+1}A_n) \mid \lambda'(A_j \dots A'_{n+1}A'_n)$ and
(b) $h(g_{A_n \dots A_{m+1}})^{-1}(g_{A_j \dots A_{n+1}})^{-1} = h'(g_{A'_n \dots A'_{m+1}})^{-1}(g_{A_j \dots A'_{n+1}})^{-1}$

Another way to view this is that $\lambda'_h(A_n \dots A_m\star) \mid \lambda'_{h'}(A'_{n'} \dots A'_{m'}\star)$ if and only if it is possible that they are adjacent in some unspecified higher level supertile.

Most importantly, we now define $\mid$ on $\{\lambda'_h(\dots A_n\star) \mid h \in G, A_n \dots A_m\star \subset \mathbb{A}, A_n \in S\}$ as
(i) $\lambda'_h(\dots A_k \dots A_n\star) \mid \lambda'_{h'}(\dots A'_k \dots A'_n\star)$ if and only if:
for all $k > n$, $\lambda'_h(A_k \dots A_n) \mid \lambda'_h(A'_k \dots A'_n)$
(ii) if $n > n'$, $\lambda'_h(\dots A_n\star) \mid \lambda'_{h'}(\dots A'_n \dots A'_{n'}\star)$ if and only if
$\lambda'_h(\dots A_n\star) \mid \lambda'_{h'(g_{A_n \dots A_{n+1}})^{-1}}(\dots A'_k \dots A'_n\star)$;
(iii) $\mid$ is symmetric.

**Remark 2.2.10** As convoluted as the definition of $\lambda'_h(\dots A_n\star) \mid \lambda'_{h'}(\dots A'_n \dots A'_{n'}\star)$ may seem, note that it effectively defines whether two infinite-level supertiles should be fitted together to make a substitution tiling (see the following Lemma and Proposition **??**).

Note that the definition ultimately rests on $\approx$, a relation discerning whether two addresses describe the same point.

As we will see in Section **??** $\approx$ and $\mid$ can be calculated automatically, in reasonably well-behaved $\Sigma(T, \sigma')$. Thus we can automatically generate the full set of substitution tilings in such a species.

**Lemma 2.2.11** *If* $\lambda'_h(\dots A_n\star) \mid \lambda'_{h_1}(\dots A^1_{n_1}\star)$ *and* $\lambda'_{h_1}(\dots A^1_{n_1}\star) \sim \lambda'_{h_2}(\dots A^2_{n_2}\star)$ *then* $\lambda'_h(\dots A_n\star) \mid \lambda'_{h_2}(\dots A^2_{n_2}\star)$
*Thus,* $\mid$ *is well-defined on the set of infinite-level supertiles.*

The proof of this lemma can be easily verified and is omitted.

**Example 2.2.12** We parallel Example **??**. Here various tilings are shown as images under $\lambda'$ and $\lambda'_h$, where $h$ is the orientation preserving isometry carrying $\mathsf{c}$ to $h\mathsf{c}$. It is probably instructive to check the labelings are correct.

Figure 7: Addressing tiles and supertiles

**Definitions 2.2.13** We close with a little more useful notation:

For all $\mathcal{A}, \mathcal{B} \in \cup_m \mathbb{A}_m^\infty$, define $\mathcal{A} \sim \mathcal{B}$ if and only if there exists an $n \in N$ such that for all $k > n$, $\mathcal{A}(k) = \mathcal{B}(k)$. Note that by Lemma **??**, for $\mathcal{A}, \mathcal{B} \in \cup_m \mathbb{A}_m^\infty$, if $\mathcal{A} \sim \mathcal{B}$ then there exist $h, h' \in G$ with $\lambda_h(\mathcal{A}) = \lambda'_{h'}(\mathcal{B})$

For all infinite-level supertiles $\mathcal{S}$, by Lemma **??**, if $\mathcal{S}$ is represented by $\lambda_h(\mathcal{A})$, $\mathcal{A} \in \mathbb{A}_n^\infty$, then for all inflated tiles $g\sigma^m A$ in $\mathcal{S}$, there exists a unique $h' \in G$, $\mathcal{B} \in \mathbb{A}_m^\infty$ with $\mathcal{A} \sim \mathcal{B}$ and $g\sigma^m A = h'\sigma^m \mathcal{B}(m)$. Define for each $\mathcal{S}$, a map

$\lambda_{\mathcal{S}} : \cup_n \mathbb{A}_n^\infty \to M$ by

for $\mathcal{B} \in \mathbb{A}_m^\infty$, $\lambda_{\mathcal{S}}(\mathcal{B}) = h'\sigma^m \mathcal{B}(m)$ and $(\lambda_{\mathcal{S}})^{-1}(g\sigma^m A) = \mathcal{B}$

This has the effect of assigning addresses to tiles in infinite-level supertiles, and is in some ways the culmination of the effort made in this section.

## 3 Addressing Substitution Tilings

For the following, we fix a real affine space $M$, a set of affine maps $G$, a similarity $\sigma$, and substitution $\sigma'$; thus substitution species $\Sigma(T,\sigma')$, prototiles $S$, substitution graph $\Gamma(T,\sigma')$, addresses $\mathcal{A}$ and labeling maps $\lambda$, $\lambda'$ and $\lambda_h$ are also fixed.

Many of the theorems in this section are well-known; however the usual proofs are often obscure, vague or ill-defined. Some of our proofs are new, all use addressing to explicitly work with supertiles, etc. The theorems at the end of this section all show that familiar properties of substitution tilings can be given simple characterizations in terms of $\Gamma(T,\sigma')$.

**Theorem 3.1.1** $\Sigma(T,\sigma')$ *is not empty.*

This follows immediately from the following lemma:

**Lemma 3.1.2** *There exists an infinite-level supertile that covers all of $M$.*

The following proof is typical of the use of addresses.

**Proof** Claim 1: For every $A \in T$, there exists a $B \in T$, $n \in \mathbb{N}$, $h \in G$ such that $hB$ lies in the interior of $(\sigma')^n(A)$

Proof of Claim 1:Let $d$ be the maximum diameter of any prototile; since the interior of $A$ is open and since $\sigma$ expands all distances, there exists an $n$ such that a ball of radius $3d$ will fit inside of $(\sigma')^n(A)$. Some tile $hB$ meets the center of such a ball, and so is contained in the interior $(\sigma')^n(A)$.

Thus for every $A$ in $T$, there is an $n \in N$, $A_n...A_0 \in \mathcal{A}_0^n$, $A_n = A$ such that

$$g_{A_n...A_1}A_0 = \lambda'(A_n...A_0\star) \subset \ int\ (\lambda'(A_n\star)) \tag{23}$$

Claim 2: There is an $A \in T$, $n \in N$, $A_n...A_0\star \in \mathcal{A}_0^n$ such that $A_n = A$, $(A_0)_T = A$ satisfying equation **??**.

Proof of Claim 2: Consider a directed graph made as follows: let the nodes of the graph represent the elements of $T$; let the edges of the graph represent addresses satisfying **??**, as follows: From each $A \in T$, for every address $A_n...A_0$ satisfying equation **??**, draw a directed edge leaving the node representing $A$ heading to the node representing $(A_0)_T$. Now this graph has at least as many directed edges as it does nodes, and so has a directed loop. Suppose these edges, in turn, represent addresses $A^1_{n_1}...A^1_0$, $A^2_{n_2}...A^2_0$,... $A^k_{n_k}...A^k_0$ and then again $A^1_{n_1}...A^1_0$, with $(A^j_0)_T = A^{j+1}_{n_{j+1}-1}$, $1 \leq j < k$, $(A^k_0)_T = A^1_{n_1-1}$. Then let

$n = n_1 + ... n_k - k$ and let $A'_n ... A'_0 \star = A^1_{n_1} ... A^1_0 \ A^2_{n_2-1} ... A^2_0 \ ... \ A^k_{n_k} ... A^k_0 \star$ It should be clear that this $A'_n ... A'_0$, $A = A'_n$ satisfies the claim.

Claim 3: Let $...A_k ... A_0 \star = ...A'_{n-1} ... A'_0 \ ...A'_{n-1} ... A'_0 \ ...A'_0 \star$. That is, $A_k = A'_{k \bmod (n)}$. Note $...A_0 \in \mathcal{A}^\infty_0$. For any $h$, we claim $\lambda_h(...A_0\star)$ is a tiling of all of $M$.

Proof of Claim 3: There exists a ball of radius some $d$ containing $\lambda'(A'_n ... A'_0\star)$ and contained in the interior of $\lambda'(A'_n\star)$. That is, there exists a ball $B$ of radius $d$ such that $\lambda'_h(A_0\star) \subset B \subset \lambda'_h(A'_n ... A'_0\star)$. By adjusting $d$, we can assume that $\lambda'_h(A_0\star)$ contains the center of $B$. Let $g \in G$ be such that $g$ takes the center of $B$ to $\mathsf{o}$, the fixed point of $\sigma$. Then $g^{-1} g^{(kn)} \sigma^{kn}(B)$ is a ball with the same center as $B$, contained in the interior of $\lambda_h(A_{kn+1} ... A_0\star)$. Thus $\lambda_h(...A_n ... A_0)$ contains a sequence of arbitrarily large, concentric balls all sharing the same center. So $\lambda_h(...A_n ... A_0)$ covers all of $M$.

□

The following relates our construction of infinite-level supertiles to the species $\Sigma(T, \sigma')$. With Theorems **??** and **??**, and Section **??**, this allows an explicit and complete description of the elements of $\Sigma(T, \sigma')$ and their orbits under $\sigma'$ (Sections **??** and **??**)

**Proposition 3.1.3** *Every tiling $\tau \in \Sigma(T, \sigma')$ is the union of a finite collection $\mathcal{S}_1, \ldots, \mathcal{S}_n$ of infinite-level supertiles such that no pair intersects in the interior of any tile in $\tau$ and for every pair $\mathcal{S}_i, \mathcal{S}_j$ that do meet, $\mathcal{S}_i \mid \mathcal{S}_j$.*

**Proof of Proposition ??**

Begin by letting $d$ be the maximum diameter of any prototile in $T$, and let $D_0$ be the ball of radius $3d$ centered at $\mathsf{o}$. For all $n \in \mathbb{N}$ let $D_n = \sigma^n(D_0)$.

Claim: For any $n \geq 0$, there is a tiling $\tau' \subset \tau$, covering $D_n$ such that $\tau'$ is the union of a finite collection $\{\lambda_{h_i}(A^i_n\star)\}$ of $n$-level where the interiors of no pair of $n$-level supertiles intersects and each pair is either disjoint or is related by $|$.

In other words, letting $\Sigma_n$ be the collection of all such tilings $\tau'$, $\Sigma_n$ is not empty.

Proof of Claim: Let $C$ be the minimal tiling in $\tau$ that covers $D_{n+1}$. By the definition of substitution tiling, there exists an $h \in G$, $A' \in T$, $j \in N$, $j > n$ such that

$$C \subset \ h(\sigma')^j(A')$$

Now, setting $A'_j = A'$,

$$h^{-1}C \subset (\sigma')^j(A') = \bigcup_{\substack{A'_j \dots A'_n \star \\ \subset A'_j \star}} \lambda(A'_j \dots A'_n \star)$$

By comments above, as in the proof of Lemma **??**, this is the union of $n$-level supertiles that do not meet in the interior of any tile. Noting that any $n$-level supertile meeting $B_n$ is contained in $C$, we thus produce a collection $\{\lambda_{h_i}(A^i_n \star)\}$ of $n$-level supertiles in $\tau$ covering $B_n$ such that no pair intersects in the interior of a tile and each pair is either disjoint or related by $|$, where each $\lambda_{h_i}(A^i_n \star) = h\lambda(A^i_j \dots A^i_n \star)$ for some $A^i_j \dots A^i_n \star \subset A'_j \star$ and $h_i = hg_{A^i_j \dots A^i_n}$. Letting $\tau'$ be the union of this collection, the claim is proven.

Suppose $\tau' \in \Sigma_n$; then note that for all $m < k$, there exists a $\tau'' \in \Sigma_m$ such that $\tau'' \subset \tau$ and if for each $n$-level supertile in $\tau'$, as given by the claim, there exists an $m$-level supertile in $\tau''$ as given by the claim. This can be seen by simply taking $\tau''$ as the union of those $m$-level supertiles in the $n$-level supertiles in $\tau'$ that meet $B_m$.

And so, there is some sequence of $\tau_k$, $\tau_k \in \Sigma_k$, $\tau_k \subset \tau_{k+1} \subset \tau$, each $\tau_k$ the union of $k$-level supertiles, as per the claim, such that the supertiles in $\tau_k$ are children of the supertiles in $\tau_{k+1}$. And so taking the appropriate unions, the Proposition is proven. □ Do yet again!

**Corollary 3.1.4** *Every tile $hA$, $h \in G$, $A \in T$ in every substitution tiling is in an infinite-level supertile $\mathcal{S}$ in $\tau$; moreover, $\mathcal{S}$ that can be represented by $\lambda_h(\dots A_0)$, where $(A_0)_T = A$ (i.e $\lambda_S(\dots A_0) = hA$).*

The corollary follows immediately from the Proposition and Lemma **??**.

The following provides a kind of converse to the preceding Proposition:

**Theorem 3.1.5** *Assume that for every $A \in T$, there exists a $h \in G$, $B \in T$, $n \in \mathbb{N}$ with $hA \in$ int $(\sigma')^n(B)$ Then every infinite-level supertile is a subset of some tiling in $\tau$.*

Note the assumption is closely related to Theorem **??**, since the assumption follows if we have: for every $A, B \in T$ there exists a $n \in N$, $h \in G$ with $hA \subset (w')^n(B)$ (by Claim 1 of Lemma **??**). Both this statement and the assumption are very easy to check for in a given example, and are usually assumed in the definition of substitution tilings.

We really do need this condition: Consider the following example: $M$ is $R^2$, $G$ consists only of translations. There are two tiles in $T$, colored squares, with

Figure 8: Outside of the domain of Theorem **??**

the substitutions shown. The infinite-level supertile pictured at right is not in any tiling in $\Sigma(T, \sigma')$.

**Proof** Let $\mathcal{S}$ be an infinite-level supertile. If $\mathcal{S}$ covers $M$ we are done. So suppose $\mathcal{S}$ does not cover $M$ and choose a tile $h'A$, $h' \in G, A \in T$ on the boundary of $\mathcal{S}$. $\mathcal{S}$ can thus be represented by $\lambda'_h(\ldots A_0)$ with $(A_0)_T = B$, $hg_{A_0})^{-1} = h'$

With no loss of generality, assume that $\mathsf{o} \in h'A$; let $D_0$ be a ball centered at $\mathsf{o}$ with radius $2d$, where $d$ is the maximum diameter of the tiles in $T$, and let $D_k = \sigma^k D_0$.

Let $\Sigma_k$ consist of all tilings of $D_k$ by $GT$ such that (i) for $\tau_k \in \Sigma_k$ there exists a supertile $(\sigma')^n(B)$ containing an image of $\tau_k$, and (ii) $\tau_k \supset (D_k \cap \mathcal{S})$. We will show that there exists a sequence of $\tau_k \in \Sigma_k$, with $\tau_k \subset \tau_{k+1}$.

Claim: each $\Sigma_k$ is non-empty

Proof of claim: By the definition of $\lambda'_h(\ldots A_0)$, there must exist an $n$ such that $\lambda'_h(A_n \ldots A_0) \supset D_k \cap \lambda'_h(\ldots A_0)$. Now there exists an $m \in \mathbb{N}$, $h'' \in G$, $B \in T$ such that $h''(A_n)_T$ is in the interior of $(\sigma')^m(B)$.

Note that $(h'')^{(n)}(\sigma')^n(A_n) \subset (\sigma')^{m+n}$ and that for $H$ ( $(h'')^{(n)}(\sigma')^n(A_n) = \lambda'_h(A_{n\ldots A_0})$,

$$H = h\ (g_{A_n \ldots A_1})^{-1}\ ((h'')^{(n)})^{-1}$$

Now choosing $m$ suitably large, we can assume that

$$H^{-1}\ D_k \subset \sigma^{m+n}(B)$$

Define then $H^{-1}\Sigma_k$ to be $(h'')^{(n)}(\sigma')^n(A_n) \cup ((\sigma')^{m+n} \cap H^{-1})\ D_k$.

The claim is proved.

Note that by the definition of each $\Sigma_k$, if $\tau \in \Sigma^k$, $\tau \cap D_{k-1} \in \Sigma_{k-1}$; thus there is a sequence $\{\tau_k \in \Sigma_k\}$, $\tau_k \subset \tau_{k+1}$, and $\cup \tau_k$ is a tiling in $\Sigma(T, \sigma')$ containing $\mathcal{S}$. $\square$

# 4 Characterizations of familiar properties

We now give characterizations of many familiar properties– unique decomposition, having connected hierarchy, cardinality, being repetitive, having self-similarity. In practice these characterizations are easy to check for in a given $\Sigma(T, \sigma')$.

See the end of Section **??** for definitions and comments on these properties.

Note that we generally have to take additional assumptions on $\Sigma(T, \sigma')$; most of these assumptions are taken most of the time by most authors, without hesitation. We chose a relatively general definition of substitution tiling, partly so that the way these assumptions were needed would be clearer.

## 4.1 Unique Decomposition

Recall that by Proposition **??**, every tiling $\tau \in \Sigma(T, \sigma')$ is the union of a finite collection $\{\mathcal{S}_1, \ldots, \mathcal{S}_n\}$ of infinite-level supertiles such that no pair intersects in the interior of any tile in $\tau$ and for every pair $\mathcal{S}_i, \mathcal{S}_j$ that do meet, $\mathcal{S}_i \mid \mathcal{S}_j$.

**Theorem 4.1.1** *If $\Sigma(T, \sigma')$ has unique decomposition, for every $\tau \in \Sigma(T, \sigma')$, for every pair of collections $\{\mathcal{S}_1, \ldots, \mathcal{S}_n\}$, $\{\mathcal{S}'_1, \ldots, \mathcal{S}'_m\}$ of infinite-level supertiles satisfying the conditions described in Proposition* **??**, *we have $m = n$ and after some reindexing, $\mathcal{S}_i = \mathcal{S}'_i$.*

Of course, if the assumptions needed for Theorem **??** are satisfied, we can conclude $\mathcal{S}_i \sim \mathcal{S}'_i$.

**Proof** Let $\tau \in \Sigma(T, \sigma')$, $\{\mathcal{S}_1, \ldots, \mathcal{S}_m\}, \{\mathcal{S}'_1, \ldots, \mathcal{S}'_m\}$ as described. Assume that there exists $\mathcal{S} \in \{\mathcal{S}_1, \ldots, \mathcal{S}_m\}$, such that $\mathcal{S} \neq \mathcal{S}'$ for all $\mathcal{S}' \in \{\mathcal{S}'_1, \ldots, \mathcal{S}'_m\}$. In particular, then, there exists a $\mathcal{S}' \in \{\mathcal{S}'_1, \ldots, \mathcal{S}'_m\}$ with $\mathcal{S}' \neq \mathcal{S}' \cap \mathcal{S} \neq \emptyset$. Now $\mathcal{S} \neq \mathcal{S}'$, so of course $\mathcal{S} \not\sim \mathcal{S}'$.

Claim: There exists $n \in \mathbb{N}$, $A_n, A'_n \in S$, $h, h' \in G$ such that $\mathcal{S}$ is represented by $\lambda'_h(\ldots A_n\star)$, $\mathcal{S}'$ is represented by $\lambda'_{h'}(\ldots A'_n\star)$, and $\mathit{int}\ h\sigma^n(A) \neq \mathit{int}\ h\sigma^n(A) \cap \mathit{int}\ h'\sigma^n(A') \neq \emptyset$.

The claim easily follows, for $\mathcal{S}$ and $\mathcal{S}'$ must agree in their non-empty intersection on all supertiles up to some level (since they agree on 0-level). But they cannot agree on supertiles on all levels or we would have $\mathcal{S} \sim \mathcal{S}'$.

Then $(\sigma)^{-n}\mathcal{S} \cap (\sigma)^{-n}\mathcal{S}' \neq \emptyset$ (as sets of points), but they cannot coexist in the same tiling in $\Sigma(T, \sigma')$— $(\sigma')^{-n}\mathcal{S} \supset h^{(-n)}A$, $(\sigma')^{-n}\mathcal{S} \supset (h')^{(-n)}A'$, $\mathit{int}\ h^{(-n)}A \neq \mathit{int}\ h^{(-n)}A \cap \mathit{int}\ (h')^{(-n)}A' \neq \emptyset$.

So there are at least two preimages of $\tau$ under $(\sigma')^{-n}$, and $\Sigma(T, \sigma')$ does not have unique decomposition. □

**Theorem 4.1.2** *Assume that for all non-trivial $h \in G$, $A, B \in T$, $hA \neq B$, $hA \neq A$. Moreover, assume that for every $A \in T$, there exists a $h \in G$, $B \in T$, $n \in \mathbb{N}$ with $hA \in \text{ int } (\sigma')^n(B)$*

*Then $\Sigma(T, \sigma')$ has unique decomposition if and only if for infinite-level supertiles $\mathcal{S}_1 = \mathcal{S}_2$ (as tilings) implies $\mathcal{S}_1 \sim \mathcal{S}_2$.*

Recall that $\mathcal{S}_1 \sim \mathcal{S}_2$ implies $\mathcal{S}_1 = \mathcal{S}_2$ by the definition of $\sim$.

The theorem is pretty sharp: there is an aperiodic example of a substitution species $\Sigma(T, \sigma')$ in which one prototile is not invariant under $G$ and different infinite-level supertiles are equivalent as tilings [**?**]. Secondly, one can easily pervert any $\Sigma(T, \sigma')$ with unique-decomposition to create a $\Sigma(T_0, \sigma'_0)$ with unique decomposition but for which $(\mathcal{S}_1 = \mathcal{S}_2) \not\Rightarrow (\mathcal{S}_1 \sim \mathcal{S}_2)$: simply add congruent copies of one or more the prototiles to $T$; these tiles will behave the same way in $\sigma'$ but produce different digits in $\mathbb{A}$. So we really do need something like the assumption stated.

Moreover, given a $\Sigma(T, \sigma')$, one can construct a new $\Sigma(T_0, \sigma'_0)$ such that every tiling in $\Sigma(T_0, \sigma'_0)$ can be decomposed into a tiling in $\Sigma(T, \sigma')$ and no tile in $T_0$ is invariant under $G$— simply mark the tiles in $T$ to destroy their symmetry (cf. Figure 16 in [**?**]). Finally, we actually only need that every pair of tiles in $T$ behaves sufficiently differently under $\sigma'$; this can be guaranteed by removing any duplicates from $T$.

The second assumption is needed to invoke Theorem **??**; there are examples in which the second assumption does not hold and the conclusion of the theorem is false.

Actually, I suspect the condition is NOT needed

The theorem is a useful technical statement. Whether or not $\mathcal{S}_1 \sim \mathcal{S}_2$ can be easily checked, as can, for two infinite supertiles, $\mathcal{S}_j, \mathcal{S}_i$, whether or not $\mathcal{S}_j \mid \mathcal{S}_i$; thus if we assume $\Sigma(T, \sigma')$ has unique decomposition, we can *explicitly* describe the entire species $\Sigma(T, \sigma')$, as in Section **??**.

**Proof** Suppose there exist infinite-level supertiles $\mathcal{S}_1, \mathcal{S}_2$ with $\mathcal{S}_1 = \mathcal{S}_2$ but $\mathcal{S}_1 \not\sim \mathcal{S}_2$. Because no element of $G$ leaves any tile $A \in T$ invariant, each tile in $\mathcal{S}_1 = \mathcal{S}_2$ can be uniquely represented as $hB$, $h \in G$, $B \in T$.

Now by Lemma **??** we can represent $\mathcal{S}_1$ by $\lambda'_{hg^{-1}_{A_0}}(\ldots A_0\star)$ and $\mathcal{S}_2$ by $\lambda'_{hg^{-1}_{A'_0}}(\ldots A'_0\star)$ where $(A_0)_T = (A_0)_T = B$, and $\ldots A_0\star, \ldots A'_0\star \subset \mathbb{A}^\infty$.
( $hg^{-1}_{A_0} A_0 = hg^{-1}_{A'_0} A'_0 = hB$. )

We must have that either $\sigma'((A_1)_T) \neq \sigma'((A'_1)_T)$ or $A_0 = A'_0$. For suppose $A_0 \neq A'_0$. Then $(A_0)^- \neq (A'_0)^-$ because this would imply the existence of some non-trivial $g \in G$ with $g\sigma'((A_1)_T) = \sigma'((A'_1)_T) = \sigma'((A_1)_T)$. But then $(A_1)_T \neq (A'_1)_T$; by our assumption, $\sigma'((A_1)_T) \neq \sigma'((A'_1)_T)$. So either $\sigma'((A_1)_T) \neq \sigma'((A'_1)_T)$ or $A_0 = A'_0$.

Now if the second case holds, either $\sigma'((A_2)_T) \neq \sigma'((A'_2)_T)$ or $A_1 = A'_1$. And, so on, and thus since $\mathcal{S}_1 \not\sim \mathcal{S}_2$, there exists an $n \geq 0$ such that $\sigma'((A_{n+1})_T) \neq \sigma'((A'_{n+1})_T)$. Take $n$ to be minimal over all choices of our initial tile in $\mathcal{S}_1 = \mathcal{S}_2$.

Without any loss of generality, we may assume $n = 0$ (if $n > 0$, note that $n$ minimal implies $\mathcal{S}_1, \mathcal{S}_2$ agree on up to $k$-level supertiles $k \leq n$, so one can uniquely apply $(\sigma')^{-n}$ to the *tilings* $\mathcal{S}_1, \mathcal{S}_2$ to obtain new $\mathcal{S}'_1, \mathcal{S}'_2$ with $\mathcal{S}'_1 = \mathcal{S}'_2$, $\mathcal{S}'_1 \not\sim \mathcal{S}'_2$ and new minimal $n = 0$).

So $\mathcal{S}_1$,$\mathcal{S}_2$ can be represented by $\lambda'_{hg^{-1}_{A_0}}(\ldots A_0\star)$ , $\lambda'_{hg^{-1}_{A'_0}}(\ldots A'_0\star)$ with $\sigma'((A_1)_T) \neq \sigma'((A'_1)_T)$; thus $(\sigma')^{-1}\mathcal{S}_1 \neq (\sigma')^{-1}\mathcal{S}_2$— they do not match as tilings.

By Lemma **??** there exists a $\tau \in \Sigma(T, \sigma')$ with $(w')^{-n}(\mathcal{S}_1) = (w')^{-n}(\mathcal{S}_2 \subset \tau)$. $\tau$ thus has two preimages in $(\sigma')^{-1}$ and so, $\Sigma(T, \sigma')$ does not have unique decomposition. □

## 4.2 Connected Hierarchy

**Theorem 4.2.1** *A tiling in $\Sigma(T, \sigma')$ has connected hierarchy if and only if every pair of tiles lies in an infinite-level supertile.*

**Proof** One direction is trivial. The other direction is proved in essentially the same way as is Proposition **??**. □

**Theorem 4.2.2** *Suppose $\Sigma(T, \sigma')$ has unique decomposition. A tiling in $\Sigma(T, \sigma')$ has connected hierarchy if and only if it is an infinite-level supertile.*

That is, the infinite-level supertiles covering $M$ are the only tilings in $\Sigma(T, \sigma')$ having connected hierarchy. There is an example, described in [**?**] of an aperiodic substitution species without unique decomposition, containing a tiling that is not an infinite-level supertile but does have connected hierarchy! The trouble is that although every pair of tiles must be within some large supertile, triples of tiles may not be within any large supertile!

**Proof**

It is clear that infinite-level supertiles covering $M$ do have connected hierarchy. Suppose a tiling $\tau \in \Sigma(T, \sigma')$ does have connected hierarchy. Then every pair of tiles lies in an infinite-level supertile, by Theorem **??**; by Theorem **??**, every tile is in a unique infinite-level supertile. Thus, one infinite-level supertile covers $\tau$. □

Figure 9: Pathological substitution graphs

## 4.3 Repetitive Tilings and species

We now characterize repetitive tilings in terms of the substitution graph $\Gamma(T, \sigma')$ (Section **??**). Let us take a moment to examine certain pathologies that can arise. In Figure **??** we see, from left to right:

(1) On the left, a very nice substitution graph: every node can be reached from every other node, by paths of every length. In fact, what matters is that every node can be reached from every other node by paths of every length greater than same given length $n$.

(2) Every node can be reached from every other node, but only by paths of certain lengths.

(3) Certain nodes cannot be reached from other nodes.

(4) The substitution graph is not even connected.

Each of these arises from a real example. Graph (1) must give a species that is strongly repetitive. Graph (2) must give a species that is not repetitive, but each tiling in the species is strongly repetitive. Graph (3) must give a species in which there is a tiling that is not repetitive at all. In graph (4), the species is essentially the union of two distinct species, each of which may or may not be repetitive.

**Theorem 4.3.1** *We must assume that for every $r \in \mathbb{R}$, there exists a finite collection of tilings covering a ball of radius $r$ in $M$, such that every tiling in $\Sigma(T, \sigma')$ is the union of images under $G$ of elements of this collection. That is we must assume that $\Sigma(T, \sigma')$ has finitely many local configurations.*

*A substitution species $\Sigma(T, \sigma')$ is strongly repetitive if there exists an $n$ such that for all $m > n$,$A, B \in T$, there exists a $h \in G$ such that $hA \subset (\sigma')^m(B)$*

*This in turn arises if and only if for every pair of nodes $A, B \in T$ in the substitution graph $\Gamma(T, \sigma')$, there exist a pair of paths $X$ and $Y$ in $\Gamma(T, \sigma')$ from $A$ to $B$, such the lengths of $X$ and $Y$ are relatively prime (or a single path of length 1).*

The assumption does not generally hold but is clearly necessary. If $\Sigma(T, \sigma')$ is "vertex-to-vertex" (see Section **??**) then the assumption does hold, and most known substitution species are vertex to vertex. Sadun [**?**] has an example of a "substitution tiling" that is not vertex-to-vertex, but requires an infinite collection of congruence classes of the tiles (but only a finite collection of similarity classes). Danzer claims to have several non-vertex-to-vertex examples but has not produced a formal proof [**?**]. At this time, there is no general finite algorithm to check whether an arbitrary $\Sigma(T, \sigma')$ has finitely many local configurations; it seems plausible that there is some algebraic characterization in terms of $\sigma$.

The condition in terms of the substitution graph $\Gamma(T, \sigma')$ is very easily checked, and so is of practical use.

**Proof** We first show that a substitution species $\Sigma(T, \sigma')$ is strongly repetitive if there exists an $n$ such that for all $m > n$,$A, B \in T$, there exists a $h \in G$ such that $hA \in (\sigma')^m(B)$. So suppose the condition holds.

Claim 1: There exists some $N \in \mathbb{N}$ such that for all $A, B \in T$, $k > N$, there exists $h \in G$ such that $hA \subset \ int\ (\sigma')^k B$

Proof of Claim 1: First, note that for every $B \in T$, there exists an $n_B \in \mathbb{N}$, $C_B \in T$, $h_B \in G$ such that $h_B C_B \subset \ int\ (\sigma')^{n_B}$ (see Claim 1 of Lemma **??**. By the hypothesis, for all $k > n$, there exists an $h$ such that $hB \subset (w')^k(h_A C_A) \subset$ $int\ (\sigma')^{k+n_A}$. Take $N = max\{n_A + n\}$.

Now fix some $r \in \mathbb{R}$, $r > 0$; by our assumption, there exists a finite collection $\mathcal{C}_r$ of tilings covering a ball of radius $2r$ in $M$, such that every tiling in $\Sigma(T, \sigma')$ is the union of elements of $\mathcal{C}_r G$. We may assume that each element of $\mathcal{C}_r$ actually appears in some tiling in $\Sigma(T, \sigma')$.

By the definition of $\Sigma(T, \sigma')$,for each $tau \in \mathcal{C}_r$, there exists a $h_\tau \in G$, $n_\tau \in \mathbb{N}$, $(A_\tau \in T$ such that $h_\tau\ \tau \subset (\sigma')^{n_\tau}(A_\tau)$.

Claim 2: There exists an $M$ such that for all $\tau \in \mathcal{C}_r$, for any $A \in T$, there exists an image of $\tau$ in $(\sigma')^M(A)$.

Proof of Claim 2: Let $M = \max\{n_\tau + N + 1\}$. Then for every $\tau \in \mathcal{C}_r$, for any $A \in T$, note that there exists an $h \in G$ with $h\ h_\tau\ \tau \subset h(\sigma')^{n_\tau}(A_\tau) \subset (\sigma')^M(A)$ since $M - n_\tau > N$, and by Claim 1.

Now we are done: Fix $r > 0$ and construct $M$ as above. Let $R$ be more than twice the maximum diameter of any $M$-level supertile. Then any tiling of radius $r$ in a tiling in $\Sigma(T, \sigma')$ will be contained in some $\mathcal{C}_r$; within every ball of radius $R$, there will be an $M$-level supertile; every $M$-level supertile contains an image of every element of $\mathcal{C}_r$ and so contains an image of every tiling of radius $r$.

The second part of the theorem should be fairly evident. The second condition really amounts to saying that there is some $N$ such that between any two nodes of the graph $\Gamma(T, \sigma')$ there exist paths of every length greater than $N$. □

**Theorem 4.3.2** *We must assume that for every $r \in \mathbb{R}$, for every tiling $\tau \in \Sigma(T, \sigma')$, there exists a finite collection of tilings covering a ball of radius $r$ in $M$, such that $\tau$ is the union of images under $G$ of elements of this collection. That is we must assume that $\tau$ has finitely many local configurations.*

*Every tiling in $\Sigma(T, \sigma')$ is strongly repetitive if there for all $A, B \in T$, there exists a $h \in G$ such that $hA \subset (\sigma')^m(B)$.*

*This occurs if and only if for every pair of nodes $A, B \in T$ in the substitution graph $\Gamma(T, \sigma')$, there exists a path $X$ in $\Gamma(T, \sigma')$ from $A$ to $B$.*

Note that this has weaker hypotheses and weaker conclusions than Theorem **??**. The proof is essentially the same as the proof above. fill in, remove certain defs, references

## 4.4 Cardinality

The following is well known; we give a simple proof. Define $\Sigma(T, \sigma')/G$ to be the equivalence classes in $\Sigma(T, \sigma')$ where two tilings are equivalent if and only if some element of $G$ takes one to the other. Then:

**Theorem 4.4.1** *The cardinality of $\Sigma(T, \sigma')/G$ is no greater than $2^{\aleph_0}$;*
*If $\Sigma(T, \sigma')/G$ has unique decomposition, the cardinality of $\Sigma(T, \sigma')/G$ is $2^{\aleph_0}$.*

**Proof** First, $\mathbb{A}_0$ has cardinality $2^{\aleph_0}$ (See Lemma **??**). Each equivalence class of $\mathbb{A}_0$ in $\sim$ is countable, and so the collection of equivalence classes is uncountable; each of these corresponds to an infinite-level supertile up to $G$, so there are $2^{\aleph_0}$ infinite-level supertiles modulo $G$. Each tiling in $\Sigma(T, \sigma')$ is the union of finitely many infinite-level supertiles; there are $2^{\aleph_0}$ finite collections of infinite-level supertiles, up to $G$, and so at most $2^{\aleph_0}$ tilings in $\Sigma(T, \sigma')$ up to $G$.

If $\Sigma(T, \sigma')$ has unique decomposition, then we note first we may modify the proof of Lemma **??** in order to find an $A \in T$, such that there exist $n, m \in \mathbb{N}$, $h, h' \in G$ such that $hA \subset \text{ int } (\sigma')^n(A)$, $h'A \subset \text{ int } (\sigma')^m(A)$ and taking $m \leq n$, $hA \not\subset (\sigma')^{n-m}(A) \subset \text{ int } (\sigma')^n(A)$. Following the proof, we may thus conclude there are $2^{\aleph_0}$ infinite-level supertiles that cover $M$. If $\Sigma(T, \sigma')$ has unique decomposition, these are all distinct as tilings and $\Sigma(T, \sigma')$ has cardinality $2^{\aleph_0}$. □

## 4.5 Self-similarity

Here we use addressing to establish a few useful theorems relating self-similar tilings to substitution species. Each substitution species contains only a few self-similar tilings, yet the self-similar tilings capture all of the local structure of tilings in the species. However, one must be very careful when extrapolating *global* properties— such as unique decomposition— from self-similar tilings to the entire species.

**Theorem 4.5.1** *For all $\tau \in \Sigma(T, \sigma')$, there exists a $\tau' \in \Sigma(T, \sigma')$ with $\sigma'(\tau') = \tau$.*

*In particular, $\sigma' : \Sigma(T, \sigma') \to \Sigma(T, \sigma')$ is onto.*

**Proof** By Proposition **??**, $\tau$ can be covered by some collection $\{\lambda'_{h_i}(\mathcal{A}_i) \mid \mathcal{A}_i \in \mathbb{A}_0^\infty\}$ that meet only along the boundaries of tiles. By Corollary **??**, there exists a collection $\{\lambda'_{h'_i}(\mathcal{A}'_i) \mid \mathcal{A}'_i \in \mathbb{A}_{-1}^\infty,\ \sigma'(\lambda'_{h'_i}(\mathcal{A}'_i)) = \lambda'_{h_i}(\mathcal{A}_i)\}$. Since, for any tiling $\tau'$ of any subset of $M$, the image $\sigma(X)$ of the set $X$ of points in boundaries of tiles in $\tau'$ is a subset of the set of points in the boundaries of tiles in $\sigma'(\tau')$, we have that the supertiles $\{\lambda'_{h'_i}(\mathcal{A}'_i)\}$ cover $M$ but meet only along the boundaries of tiles.

So $\tau' = \cup_i \lambda'_{h'_i}(\mathcal{A}'_i)$ is a tiling of $M$ with $\sigma'(\tau') = \tau$. □

**Theorem 4.5.2** *For any $\Sigma(T, \sigma')$ there exists an $n \in \mathbb{N}$, $\tau \in \Sigma(T, \sigma')$ with $(\sigma')^n(\tau) = \tau$. In particular, every substitution species contains a self-similar tiling.*

**Proof** Really this is a corollary of Lemma **??**. Recall that in the proof of that Lemma, we found a tile $A \in T$ such that there existed an $n \in N$, $h \in G$ with $hA \subset$ *int* $( (\sigma)'(A) )$; thus $h = g_{A_n \ldots A_0}$ for some $A_n \ldots A_0 \star \in \mathbb{A}_0^n$ with $A_n = A = (A_0)_T$. Let $X$ be the string $A_{n-1} \ldots A_0$.

Define $\mathcal{A} \in \mathbb{A}_0^\infty$ with $\mathcal{A} = \overline{X}\bullet$.

Similarly define $\mathcal{B} \in \mathbb{A}_{-\infty}^0$ with $\mathcal{B} = A_0 \bullet \overline{X}$.

Now let $x = \lambda(\mathcal{B}) \in A$. Let $h' \in G$ take $x$ to $\circ$. Then we have, as can easily be verified (See Lemmas **??** and **??** :

$$(\sigma')^n(\lambda_{h'}(\mathcal{A})) = \lambda_{h'}(\mathcal{A}) \in \Sigma(T, \sigma')$$

□

**Theorem 4.5.3** *If $\Sigma(T, \sigma')$ has unique decomposition, then for any $n \in \mathbb{N}$, there exists $\tau \in \Sigma(T, \sigma')$, $m > n$ with $(\sigma')^m(\tau) = \tau$ and for all $k < m$, $(\sigma')^k(\tau) \neq \tau$*

**Proof** This is proved very much in the style of Lemma **??**. Fix $n \in \mathbb{N}$.

We first show that for every $A \in T$, there exists a $B \in T$, $j \in \mathbb{N}$, $j > n$, $g_{A_j \ldots A_0} \in G$, $A_n = A$, $(A_0)_T = B$ such that $g_{A_j \ldots A_0} B$ lies in the interior of $(\sigma')^j(A)$ but $g_{A_j \ldots A_2} \sigma'(A_1)$ meets the boundary of $(\sigma')^j(A)$. (Note that $g_{A_j \ldots A_2} \sigma'(A_1) \supset g_{A_j \ldots A_1} A_0 = g_{A_j \ldots A_0} B$).

Begin by choosing a tile $hB$ in $(\sigma')^n(A)$ meeting the boundary of $(\sigma')^n(A)$. By Claim 1 of the proof of Lemma **??** there exists a $j \in \mathbb{N}$, $j > n$, and

By Lemma **??** $hB = \lambda(A_n \ldots A_0 \star)$ for some $A_n \ldots A_0 \star \subset A_n \star$ where $A_n = A$ and $(A_0)_T = B$. Check

Simply let $d$ be the maximum diameter of any prototile; since the interior of $A$ is open and since $\sigma$ expands all distances, there exists an $n$ such that a ball of radius $3d$ will fit inside of $(\sigma')^n(A)$. Some tile $hB$ meets the center of such a ball, and so is contained in the interior $(\sigma')^n(A)$.

Thus for every $A$ in $T$, there is an $n \in N$, $A_n ... A_0 \in \mathcal{A}_0^n$, $A_n = A$ such that

$$g_{A_n \ldots A_1} A_0 = \lambda'(A_n ... A_0 \star) \subset \; int \; (\lambda'(A_n \star)) \tag{24}$$

Claim 1: There is an $A \in T$, $n \in N$, $A_n ... A_0 \star \in \mathcal{A}_0^n$ such that $A_n = A$, $(A_0)_T = A$ satisfying equation **??**. FINISH!

□

**Corollary 4.5.4** *If $\Sigma(T, \sigma')$ has unique decomposition, $\Sigma(T, \sigma')$ contains a countably infinite collection of self-similar tilings.*

The proof is immediate. Contrast this with Theorem **??** below.

The following provides a nice contrast to the above theorems. Something like the following should be true in virtually any setting:

**Theorem 4.5.5** *For all $n \in N$, for $M = \mathbb{R}^{\dim}$, $G$ a group of measure-preserving affine maps that includes translations, there exists a substitution species $\Sigma(T, \sigma')$ such that for any $0 < k < n$, for any $\tau \in \Sigma(T, \sigma')$, $(\sigma')^k(\tau) \neq \tau$*

**Proof** For any $n, \dim \in \mathbb{N}$, we produce a virtually trivial example in $\mathbb{R}^{\dim}$. Simply define a tile $A = [-1, 1]^{\dim}$, the unit hypercube. Define $\sigma$ to be the similarity that simply expands all coordinates by a factor of $2^{1/n}$. Let $T =$

$\{\sigma^k A \mid k \in \mathbb{Z}, 0 \leq k < n\}$, and define for each $\sigma^k A \in T$ with $k < n-1$, $\sigma'(\sigma^k A) = \sigma^{k+1} A$. Define $\sigma'(\sigma^{n-1} A) = \cup(A + (\pm 1, \ldots, \pm 1))$, the natural tiling of $[-2,2]^{\dim}$ by $2^{\dim}$ translated copies of $A$.

It should be clear that the elements of $\Sigma(T, \sigma')$ are each square lattice tilings and that for any $0 < k < n$, for any $\tau \in \Sigma(T, \sigma')$, $(\sigma')^k(\tau) \neq \tau$. □

**Corollary 4.5.6** *(to Theorem* **??***): A infinite-level supertile $\mathcal{S}$ is self-similar if and only if there is a $\mathcal{A} \in \mathbb{A}^{\infty}_{-\infty}$ of the form $\overline{X} \bullet \overline{X}$ where $X$ is some finite string, such that $\mathcal{S}$ can be represented by $\lambda_h(\overline{X}\bullet)$ and $A_0)$ with $h \in G$ so $\mathsf{o} = h\lambda(\bullet\overline{X})$*

**Proof** Suppose $\mathcal{S}$ is self-similar; thus there exists an $n \in \mathbb{N}$ with $(\sigma')^n(\mathcal{S}) = \mathcal{S}$

Note we must have $\mathsf{o} \in \mathcal{S}$. Let $\mathsf{o} \in hA \subset \mathcal{S}$ for some tile $hA$. Then by Lemma **??** there exists some $A_0 \ldots \in \mathbb{A}^0_{-\infty}$ with $h\lambda(A_0 \ldots) = \mathsf{o}$. Also, there exists an $\ldots A_0 \in \mathbb{A}^{\infty}_0$ with $\mathcal{S} = \lambda'_h(\ldots A_0)$. Applying Lemma **??** and Lemma **??** we see that for all $0 < k < n$ we must have $A_k = A_{k-n}$ and indeed, for all $k \in \mathbb{N}$, $m \in \mathbb{Z}$, $A_k = A_{k+mn}$. Let $\mathcal{A} \in \mathbb{A}^{\infty}_{-\infty}$ be defined by $\mathcal{A}(k) = A_k$, and $X = A_{n-1} \ldots A_0$ and we are done.

The converse is immediate. □

**Corollary 4.5.7** *There are at most countably infinite self-similar tilings in $\Sigma(T, \sigma')$. In particular, if $\Sigma(T, \sigma')$ has unique decomposition there is a countably infinite collection of self-similar tilings in $\Sigma(T, \sigma')$.*

Contrast this to Theorem **??**. On the other hand, one can easily show that every bounded neighborhood in a tiling in $\Sigma(T, \sigma')$ appears in a self-similar tiling. Still, one must be careful when extrapolating properties from self-similar tilings to entire substitution species.

IS this REALLY true, or just a peice of folk-lore?

**Theorem 4.5.8** *A tiling is self-similar if and only if it is the union, with disjoint interiors, of a finite collection of infinite-level supertiles, which are pairwise disjoint or related by $|$.*

**Proof** □

$|S|^n$ in $|S|^n$

# 5 Three applications of addressing

At long last we give some examples of how addresses might be used to prove new theorems.

## 5.1 Automatic descriptions of adjacencies

Roughly speaking, a set of strings is "automatic" if the set can be generated by some algorithm; an equivalence relation on a set of strings is automatic if it can be checked by some algorithm.

We turn to automatic descriptions of structures in $\Sigma(T, \sigma')$. The third part of the following definition is somewhat non-standard, but only because we are struggling to fix a "decimal point" in automatic infinite strings.

**Definition 5.1.1** A set $A$ of *finite strings* of digits drawn from some finite alphabet $D$ is **automatic** if and only if there exists a finite directed graph $\Gamma$ with nodes in exact correspondence with $D$, such that $A$ exactly consists of strings corresponding to directed paths in $\Gamma$.

A set $A$ of *infinite strings* of digits drawn from some finite alphabet $D$ is **automatic** if and only if the set of all finite substrings of strings in $A$ is automatic (that is, if and only if there exists a finite directed graph with nodes in exact correspondence with $D$, such that $A$ exactly consists of strings corresponding to infinite directed paths in $\Gamma$.

A *set $A$ of maps* $\mathbb{Z}$ to some finite alphabet $D$ is **automatic** if and only if there exists a finite directed graph with nodes in exact correspondence by with $D$, such that each map in $A$ corresponds with an infinite directed path in $\Gamma$, and each infinite directed path in $\Gamma$ corresponds to a collection $\{\mathcal{A}, \varsigma^n(\mathcal{A}) \mid n \in \mathbb{Z}\} \subset A$.

The graph $\gamma$ is a **finite-state automaton**.

Recall Definition **??**:

**Theorem 5.1.2** *For any set of addresses* $\mathbb{A}$*,* $\mathbb{A}$ *is automatic.*

**Proof** We must provide the automaton, denoted $\Gamma(\mathbb{A})$. Take nodes labeled in $T \cup S \cup \{\oslash\}$. From every node $A$ in $T \cup S$, draw a directed edge leaving $A$ to each $B \in A^+ \subset S$. Draw a directed edge from $\oslash$ to each node in $T \cup S \cup \{\oslash\}$.

It should be clear that the infinite paths in $\Gamma(\mathbb{A})$ are in correspondence with the addresses $\mathbb{A}$. □

In fact, we can simplify our lives if we pretend that the substitution graph

$\Gamma(T,\sigma')$ is the automaton required: addresses correspond to infinite paths in $\Gamma(T,\sigma')$ (that may or may not have a starting point but do go on forever), where the path is somehow parametrized by $\mathbb{Z}$ (and note too that the digits in an address correspond to edges in $\Gamma(T,\sigma')$ but nodes in $\Gamma(\mathbb{A})$).

**Definition 5.1.3** A relation $\cong$ on an automatic set of strings of digits in $D$ is automatic if and only if there exists a directed graph $\Gamma$ with edges labeled in $D \times D$ such that paths in $\Gamma$ exactly correspond to pairs of strings related by $\cong$.

A relation $\cong$ on an automatic set of maps $\mathbb{Z} \to D$ is automatic if and only if there exists a directed graph $\Gamma$ with edges labeled in $D \times D$ such that paths in $\Gamma$ exactly correspond to equivalence classes under the shift map $\varsigma$ of pairs of strings related by $\cong$.

Effectively, the above is a description of a Mealy machine. We extend the definition to addresses. Again, the decimal representation of $\mathbb{R}$ motivates the definition.

**Theorem 5.1.4** *If $\Sigma(T,\sigma')$ has finitely many local configurations, then $\approx$ on $\mathbb{A}$ is automatic.*

This theorem is rich in corollaries. Recall $\approx$ is the basis for defining $|$.

**Proof** There are finitely many ways a tile may meet its neighbors; let these be given by $\{(A,hB)\} \in T \times GT$. Construct a graph $\Gamma'$ as follows: the nodes are in exact correspondence with the $(A,hB)$; for every $(A,hB)$, for every $C \in A^+ \subset S$, $D \in B^+ \subset S$ such that $C \cap h^{(1)}D \neq \emptyset$, draw a directed edge leaving $(A,hB)$, labeled $(C,D)$ and arriving at $(C_T,(g_C)^{-1}\ h^{(1)}g_D D_T) \in \{(A,hB)\}$.

Now in some essential sense, $\Gamma'$ is the graph we care about, just as $\Gamma(T,\sigma')$ really encodes the automatic structure of $\mathbb{A}$. But to be careful, we must go ahead and construct another graph to account for $\oslash$ and the breaking of single tiles into sets of pairs of neighboring tiles by $\sigma'$.

So construct $\Gamma(\approx)$ as follows: Simply begin with a copy each of $\Gamma'$ and $\Gamma(T,\sigma')$, with one additional node labeled $\oslash$. Recall the edges in $\Gamma(T,\sigma')$ exactly correspond to elements of $S$. Label each edge in the copy of $\Gamma(T,\sigma')$ by $(A,A)$, where $A \in S$ corresponds to the edge. For each node in $\Gamma(\mathbb{A})$ labeled $A \in S$, for each pair of tiles $B,C \in A^+ \subset S$ that are not disjoint in $\sigma'(A)$, draw edges from $A$ in $\Gamma(\mathbb{A})$ to the nodes $(B_T,(g_B)^{-1}\ g_C C_T)$ and $(C_T,(g_C)^{-1}\ g_B B_T)$; label these edges $(B,C)$ and $(C,B)$, respectively. Finally, draw edges from the node $\oslash$ to itself and to the nodes in the copy of $\Gamma(T,\sigma')$; label each of these $(\oslash,\oslash)$

It should be clear that $\mathcal{A},\mathcal{B} \in \cup_n \mathbb{A}^n$ correspond to equivalent points in $M$ if and only if they correspond to a path in $\Gamma$ such that for each $n \in \mathbb{Z}$, the $n$-th edge in the path is labeled $(\mathcal{A}(n),\mathcal{B}(n))$, $\square$

**Remarks 5.1.5** This theorem seems very useful in a variety of ways. In the first place, it allows us to explicitly check whether two addresses give the same point, and consequently, whether two addresses in $\mathbb{A}_m$ give adjacent tiles, supertiles or infinite-level supertiles.

**Theorem 5.1.6** *There is an algorithm that will terminate if and only if* $\Sigma(T, \sigma')$ *has finitely many local configurations. If* $\Sigma(T, \sigma')$ *does have finitely many configurations, this algorithm will produce the nodes of the graph* $\Gamma$ *in the preceding proof.*

The point of this Theorem is that, in practice it is easy to compute, by hand or on a computer, the pairs of adjacent tiles needed in the preceding construction— *if* $\Sigma(T, \sigma')$ does have finitely many local configurations.

**Proof** Simply iterate the following procedure:

(1) Let $Q_0$ be the collection of pairs $(B_T, (g_B)^{-1} g_C C_T)$ where $B, C \in A^+$ for some $A \in T$ and $B \cap C \neq \emptyset$.

(2) Suppose $Q_i$ is defined and non-empty for all $i < k$. Let $Q_k$ be the collection of pairs $(B_T, (g_B)^{-1}\, h\, g_C C_T)$ where (i) for all $i < k$, $(B_T, (g_B)^{-1}\, h\, g_C C_T) \notin C_i$ and (ii) there exists a $(A, hD) \in Q_{k-1}$ with $B \in A^+$, $C \in D^+$ and $B \cap h^{(1)} C \neq \emptyset$.

(3) If $Q_k = \emptyset$, then we are done and $\cup_{i<k}\, Q_i$ is the desired collection of adjacencies. Otherwise, return to step (2).

Not the procedure terminates if and only if there are finitely many local configurations. □

**Example 5.1.7** Examples are certainly in order. We will give two; in precisely the manner described above, the author has computed the adjacencies for the $L$-tiling and the tilings of the line arising from the symbolic dynamical system $0 \to 1$, $1 \to 10$.

Figure 10: A simple example

We begin with a simple example first. The space is $\mathbb{R}$ with the usual metric, $G$ consists of translations. The tiles are segments of length 1 and $\phi = \frac{1}{2}(1+\sqrt{5})$, denoted $\mathsf{0}$ and $\mathsf{1}$ respectively. The inflation $\sigma$ is a similarity of modulus $\phi$. $\sigma'(\mathsf{0}) = \mathsf{1}$, $\sigma'(\mathsf{1}) = \mathsf{1} \cup (\mathsf{0} + \phi)$. We denote the image of $\mathsf{1}$ in $\sigma'(\mathsf{0})$ by $\mathsf{a}$, the image of $\mathsf{1}$ in $\sigma'(\mathsf{1})$ by $\mathsf{b}$ and the image of $\mathsf{0}$ in $\sigma'(\mathsf{1})$ by $\mathsf{c}$. Thus $T = \{\mathsf{0}, \mathsf{1}\}$, $S = \{\mathsf{a}, \mathsf{b}, \mathsf{c}\}$, as indicated at the top of Figure **??**.

On the middle left in Figure **??**, the substitution graph is shown; on the middle right we see $\Gamma(\mathbb{A})$

On the bottom left we see the graph $\Gamma'$ and finally on the bottom right we have $\Gamma(\approx)$. It is now apparent that $\mathcal{A} \approx \mathcal{B}$ if and only if $\mathcal{A}, \mathcal{B}$ are identical or of the form

$$\mathcal{A} = \#\mathsf{ca}\overline{\mathsf{bb}}$$
$$\mathcal{B} = \#\mathsf{bc}\overline{\mathsf{ac}}$$

where $\#$ is some string common to both $\mathcal{A}$ and $\mathcal{B}$, ending in either $\mathsf{a}$ or $\mathsf{b}$

We now give a more complex example: the $L$-tiles of Figure 1: After some calculation, one arrives at the graph $\Gamma(\approx)$ indicated in Figure **??** (Since there is only one element of $T$, we supress the nodes coming from $\Gamma(\mathbb{A})$; however, the directed edges from these nodes are important and have been included. The dark arrows each represent a group of arrows from each node labeled in $T \cup S$.)

Figure 11: A complex example

And, though the rules are more complex to write out, it is not hard to check that, for example,

$$\#\mathsf{acbbbc}\overline{\mathsf{b}}$$
$$\approx \#\mathsf{caaacc}\overline{\mathsf{b}}$$

as indicated in the bottom right of Figure **??**.

**Lemma 5.1.8** $\mathbb{A}_{-\infty}^{\infty}$ *is automatic;* $\sim$ *is automatic on* $\mathbb{A}_{-\infty}^{\infty}$

**Proof** This is really trivial: to show $\mathbb{A}_{-\infty}^{\infty}$ is automatic, use the graph formed by deleting the node $\oslash$ from $\Gamma(\mathbb{A})$. To show $\sim$ is automatic, form the graph $\Gamma(\sim)$ as follows: let the nodes be in correspondence with $S \cup \{\star\}$; from each node corresponding to, say, $A \in S$, draw two edges labeled $(A, A)$, one returning to the node itself, the other going to the node labeled $\star$. Draw $|S|^2$ edges from $\star$ to itself, labeled in $S \times S$. □

**Corollary 5.1.9** | *is automatic on the sets* $\cup \mathbb{A}_m^n$ *and*

It really is easy to check whether addresses give adjacent tiles. It is also possible to construct automata that quickly produce, say, all addresses adjacent to a given address, etc.

**Application of the Application**

There are many numerical simulations or computations for which a grid is necessary: modeling fluid flow, finding algebraic curves implicitly, studying the dynamics of a population, and literally hundreds of other examples.

For our purposes, these fall into two classes: those in which highly regular, self-similar structure is desired, and those for which highly regular structure is very much unwanted.

Finding the solutions to a set of algebraic equations numerically is a perfect example of the first class. One divides the space into some sort of tiling and examines each tile for the presence of part of the solution set. If the set of solutions appears to intersect the tile, the tile is subdivided and each smaller tile is examined, etc. Clearly this is closely connected to the study of self-similar tilings, although only a handful of substitutions are used in practice (those related to the cubic lattice). However, one problem does arise that we shed light on: how does one systematically keep track of all those little tiles—their positions, their adjacencies, and so forth. The brute-force solution is to keep detailed information about every tile created. The simpler solution is to simply keep each tile's address, from which the tiles position and adjacencies can readily be calculated. (And in Section **??** we will see that we can do with even less).

However, often highly regular grids are exactly what one does not want; the geometry of the grid may give misleading results. As a simple example, imagine measuring distance in the Euclidean plane, but by measuring along the edges of an arbitrarily fine square lattice. No matter how fine the lattice, of course, one's answer is likely to be dramatically off. On the other hand, highly irregular grids would appear to have a high cost in storage needs. It would appear at first glance that detailed information would have to be kept about the position, shape and adjacencies of every cell.

But self-similar grids *can* appear highly irregular. Radin and Sadun [**?**] have shown, for example, that distance can be measured to arbitrary accuracy along the edges of the pinwheel tilings (and indeed, this key property is likely to be generic in some sense). We can easily loosen our restrictions on $G$ and obtain some crazy-looking, highly amorphous substitution tilings that still have all the structure outlined in this paper. Thus, one might have the best of both worlds– a highly regular combinatorial structure that eases certain computational tasks, but is less sensitive to the geometry of the problem at hand.

Over the next few years, this author would like to work with various researchers involved with computational modeling to see if addressing in substitution tilings is in fact practical.

### 5.2 Explicit descriptions of $\Sigma(T, \sigma')$

### 5.3 Orbits in $\Sigma(T, \sigma')$ under $\sigma'$

Explicitly describing orbits in $\Sigma(T, \sigma')$ under $\sigma'$ is a perfect example of how addressing can be exploited. For these examples, we will need a bit of the structure outlined in Section **??**.

## 6 Locally-finite encoding of addresses

Addresses provide concrete understanding of substitution tilings, at least for this author. Recently, I constructed matching rules for all known substitution tilings [**?**]; that is, given a substitution species, I gave an algorithm that produces a set of marked tiles such that, when they are put together, they must recreate the tilings in the original substitution species.

Part of the problem comes down to this: if every tile in a substitution tiling is meant to play a role in an infinite hierarchy (i.e. be assigned an address $\dots A_n \dots A_0 \star$), how can this be done with a *finite* collection of markings. In other words, how can one encode an infinite amount of information in a locally finite manner? (There are really two steps to the proof: first, one must cook up the correct set of markings, by examining $\Sigma(T, \sigma')$; secondly one must show that any tiling made with the new marked tiles is more or less equivalent to a tiling in $\Sigma(T, \sigma')$. In the second part of the proof, one can view the markings as a "wish list", denoting a tile's intended role in an infinite level supertile.)

I would like to describe the initial ideas behind the construction of the markings in [**?**]. As it turned out, these ideas turned out to be technically redundant and have vanished without trace from [**?**]; this is unfortunate, since they provide the clearest understanding of how the construction works. We won't present a series of theorems and lemmas, but will instead give an expository discussion.

Recall that in an infinite-level supertile can be regarded as a tiling by $\lambda'_h(\dots A_0\star)$; we could easily define formally a map from $A_0^\infty$ to the set of tiles in infinite-level supertiles (modulo $G$) such that $\mathcal{A}$, $\mathcal{B}$ correspond to tiles in the same infinite-level supertile if and only if $\mathcal{A} \sim \mathcal{B}$; more generally, we can associate $A_n^\infty$ with $n$-level supertiles, with all the structure developed in Section **??**.

The problem before us is, how can one mark the tiles with some finite collection of symbols such that the addresses $\mathcal{A}, \mathcal{B}$, etc are apparent?

We begin by recalling that $S$ must have at least two elements; with a little finessing, we may more assume that in fact, for all $A \in S$, $A^+$ has at least two elements. For each $A$, choose two special elements of $A^+$— a "local key" and a "regional key". Note that every element of $S$ is exactly one of": a local key, a regional key or neither.

We will mark each *supertile* in a substitution tiling with an ordered pair of digits drawn from $S \cup \oslash$. The first digit in the pair will be called the tile's "primary register"; the second digit in the pair will be called the tile's "secondary register". Every supertile has some address $...A_n\star$, where the supertile is of level $n$ (recall tiles are 0-level supertiles). The primary register of each supertile will contain the digit $A_n \in S$.

The idea is that using the secondary registers, the markings on the tiles alone can encode, without ambiguity, the primary registers of all higher level supertiles, and so encode the address of the entire infinite level supertile.

So, for every $n$-level supertile $\dots ...A_n\star$:

If $A_{n\in S}$ is a local key, the secondary register is to contain the contents $(A_{n+1})$ of the primary register of the supertiles' parent.

If $A_{n\in S}$ is a regional key, the secondary register is to contain the contents of the secondary register of the supertiles' parent.

If $A_{n\in S}$ is not a key, the secondary register is to contain null symbol $\oslash$.

Claim 1: every tile (except, possibly, one special tile) in an infinite-level supertile will be unambiguously marked by the above procedure.

Claim 2: Once the tiles are marked, the address of any tile can be read off from the markings (once we figure out how to compare information over arbitrary distances).

The claims are virtually tautological.

Proof of Claim 1: Take any infinite-level supertile; every tile, with at most one exception, will be have an address $\dots A_0$ of one of the following forms: $\dots N$, $\dots NL$, $\dots LL$, $\dots NRR \dots R$, or $\dots LR \dots R$ (where $L$ denotes the digit is a local key, $R$ denotes the digit is a regional key, and $N$ denotes the digit is not a key).

In each case the contents of the primary register will be $A_0$; in all but the third and fifth cases the secondary register will be $\oslash$. In the third case, the secondary register will contain the digit $A_2$. In the last case the secondary register will contain the digit $A_{n+1}$, where $A_n$ is a local key and $A_k$ is a regional key for all $k < n$.

There may be a single tile with an address of the form $\ldots R \ldots R$. It does not matter what the contents of the secondary register are; for various reasons, the techniques in [**?**] could never make use of such a marking.

Proof of Claim 2: To find the address of any tile in an infinite-level supertile, we reconstruct the primary registers of every supertile containing the tile. This is easy: If a supertile has address $\ldots A_n\star$, we find the contents of the supertiles primary and secondary registers in the secondary registers of $\ldots A_n L\star$ and $\ldots A_n R\star$, respectively. By induction, it is clear that these digits are marked in the secondary registers of the tiles $\ldots A_n LR \ldots R\star$ and $\ldots A_n RR \ldots R\star$, respectively.

So this is how information can be stored. The remaining trouble

# 7 Philosophical epilogue

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