Addressing in substitution tilings
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1. Definitions

Examples will follow in Section . The most general setting is much broader than that taken here, but for the sake of establishing ideas, we will work in $\mathbb{R}^d$, real affine space, endowed with the usual metric and usual measure μ; we let G be any group of measure-preserving affine transformations on $\mathbb{R}^d$. with identity e.

A **tile** is a compact, measurable set in $\mathbb{R}^d$. We restrict ourselves to using tiles with non-empty interior[1]. A **configuration** is a collection of tiles with disjoint interiors in $\mathbb{R}^d$. For a configuration τ, the **support** $[\![\tau]\!]$ of τ is defined as $[\![\tau]\!] := \bigcup_{A\in\tau} A$. A **tiling** is a configuration with support $\mathbb{R}^d$. A **species** is a collection of tilings.

We adopt the convention, for any configurations τ_1, τ_2, that $\tau_1 \cup \tau_2$ is defined only if $\tau_1 \cup \tau_2$ is also a configuration (all tiles in $\tau_1 \cup \tau_2$ have disjoint interiors). Note that $\tau_1 \cap \tau_2 = \emptyset$ if and only if $[\![\tau_1]\!] \cap [\![\tau_2]\!] \subset \partial[\![\tau_1]\!] \cap \partial[\![\tau_2]\!]$ markings

Next, let σ be an **inflation**, any expanding affine map with unique fixed point o, such that the cyclic group generated by σ is normal in $< \sigma, G >$, the subgroup of all affine transformations from $\mathbb{R}^d$ to $\mathbb{R}^d$ generated by elements of G and σ. We'll call o the origin[2].

For all $g \in G$, for any integer n, there exists $h \in G$ with $\sigma^n g = h\sigma^n$. We define for each $g \in G$, a family $\{g^{(n)} \mid n \in \mathbb{Z}\} \subset G$ such that $g^{(0)} = g$ and $\sigma^n g = g^{(n)}\sigma^n$. It soon follows that, for all $n, m \in \mathbb{Z}$: $\sigma^n g^{(m)} = g^{(n+m)}\sigma^n$; $(g^{(m)})^{(n)} = g^{(n+m)} = (g^{(n)})^{(m)}$ and $(g^m)^{(n)} = (g^{(n)})^m$

Let T be some finite set of tiles; the elements of T will be called **prototiles**. With no loss of generality, we require that all the elements of $\{\sigma^n(A) \mid A \in T, n \in \mathbb{Z}\}$. For a tile $B = gA$, $g \in G$, $A \in T$, we will denote $B_T := A$, and $g_B := g$. Thus $B = g_B B_T$, $B_T \in T$ Let $C(T)$ be the set of configurations of the form $\{g_i A_i\}$, each $g_i \in G$, each $A_i \in T$.

A **substitution** is a map $\sigma' : C(T) \to C(T)$ such that for all $\tau \in C(T)$, $g \in G$,
(i) $\sigma'(\tau)$ is a configuration with $[\![\sigma'(\tau)]\!] = \sigma([\![\tau]\!])$.
(ii) $\sigma'(g\tau) = g^{(1)}\sigma'(\tau)$
(iii) For $\tau_0 \subset \tau$, $\sigma'(\tau_0) \subset \sigma'(\tau)$.

Note then that σ' is determined by its action on the prototiles T, and that for τ_1, $\tau_2 \in C(T)$, we have $\sigma'(\tau_1) \cup \sigma'(\tau_2) = \sigma'(\tau_1 \cup \tau_2)$.

Given T and σ', an **n-level supertile** is any configuration of the form $(\sigma')^n(A)$, or $(\sigma')^n(gA)$,

[1] We need non-empty interior to ensure that $\Sigma(T, \sigma')$ is non-empty.
[2] Note that the existence of such a σ constrains G quite a bit (In particular, no such G exists if G is the full group of measure preserving affine transformations).

$A \in T$, $g \in G$, $n \in \mathbb{N}$. Note that our these are well defined by our conditions on σ'.

The **substitution species** $\Sigma(T, \sigma')$ of substitution tilings arising from T, σ' is the collection of tilings in $C(T)$ such that $\tau \in \Sigma(T, \sigma')$ if and only if for every $\tau' \subset \tau$ with bounded support, there exists a supertile $(\sigma')^n(gA)$ such that $\tau' \subset (\sigma')^n(gA)$.

$\Sigma(T, \sigma')$ has **unique decomposition** if and only if $\sigma' : \Sigma(T, \sigma') \to \Sigma(T, \sigma')$ is one-to-one.

For each $A \in T$, $\sigma'(A)$ is a configuration; we will find it useful to denote the set of tiles in $\sigma'(A)$ as A^+[3] and let S be the set of tiles given by

$$S := \bigcup_{A \in T} A^+$$

Note that S is partitioned into the A^+ (this is the sole reason we required that the elements of T be disjoint).

For all $A \in S$, denote $A^+ := (A_T)^+$

For all $A \in S$, there is a unique $B \in T$ with $A \in B^+$. Let $A^- := B$

For all $A \in T$, let $A^- := \{B^- \in T \mid B \in S, B_T = A\}$.

Note S is finite (since the elements of T have non-zero measure and so each $\sigma'(A), A \in T$ consists of finitely many tiles). Index the elements of S so that $S = \{S_i\}$ and define the $|S| \times |S|$ matrix $\mathbf{S}$ by

$$\mathbf{S}_{ij} := \begin{cases} 1 & \text{if } S_j \in (S_i)^+ \\ 0 & \text{otherwise} \end{cases}$$

Define **addresses** $\mathbb{A} = \Sigma_{\mathbf{S}} \subset S^{\mathbb{Z}}$, the two sided subshift of finite type, with alphabet S defined by the matrix $\mathbf{S}$. It will also be convenient to associate a **substitution graph** $\Gamma(T, \sigma')$ with $\mathbb{A}$; this graph will have vertices indexed by T and directed edges indexed by S; an edge, indexed A, leaves a vertex X for vertex Y if and only if $Y = A_T$, $A \in T^+$. Thus the elements of $\mathbb{A}$ exactly correspond to bi-infinite paths of edges in $\Gamma(T, \sigma')$.

Let ς be the **shift** acting on $\Sigma_{\mathbf{S}}$ by $\varsigma(\alpha) = \beta$ iff $\alpha_n = \beta_{n+1}$ for all $n \in Z$.

We will take **blocks** $[A_n \ldots A_m] := \{\alpha \in \mathbb{A} \mid \alpha_j = A_j \text{ for } n \geq j \geq m\}$ as a basis for a topology on $\mathbb{A}$. For each $\alpha \in \mathbb{A}$ it will be useful to abuse notation and define $[\alpha] := \{\beta \in \mathbb{A} \mid \exists n \in \mathbb{Z} \text{ s.t. } \forall k \geq n, \beta_k = \alpha_k\}$. Note $\mathbb{A}$ is partitioned into these $[\alpha]$.

For all $n \in \mathbb{Z}$, let $\mathbb{A}^n = \{[A_n \ldots]\}$, the set of all infinite-to-the-right blocks with first index n. Let $\varsigma : \mathbb{A}^n \to \mathbb{A}^{n+1}$ in the obvious manner.

For any block $[A_n \ldots A_m] \subset \mathbb{A}$, define $g_{[A_n \ldots A_m]} := g^{(n)}_{A_n} \ldots g^{(j)}_{A_j}$. Now define the map $\lambda : \mathbb{A}^n \to \mathbb{R}^d$ by[4]

$$\lambda([A_n \ldots]) := \bigcap_{j \leq n} g_{[A_n \ldots A_j]} \sigma^j (A_j)_T$$

Now we should note the following:

Lemma 1.1 *1. $\lambda : \mathbb{A}^n \to \mathbb{R}^d$ is well defined.*

2. $\lambda(\varsigma[A_n \ldots]) = \sigma\lambda([A_n \ldots])$.

3. $\lambda([A_n]) = \sigma^n(A_n)$ (Note $[A_n] = \bigcup_{[B_n \ldots] \in \mathbb{A}^n, B_n = A_n} [B_n \ldots]$

[3]That is, $A^+ := \sigma'(A)$, but we regard A^+ as a set of symbols and $\sigma'(A)$ as a configuration of tiles.

[4]denoted by λ since we are λabeling points

4. $\lambda([A_n \ldots A_0]) = g_{[A_n \ldots A_0]}(A_0)_T = g_{[A_n \ldots A_1]} A_0$

5. For $n \in \mathbb{N}$, $A \in S$, *the supertile* $(\sigma')^n(A) = \{\lambda([A_n \ldots A_0.]) \mid A_n = A\}$

Proof (1) That $\lambda([A_n \ldots])$ is a single, well-defined point in $\mathbb{R}^d$ follows from the observation that the sets $g_{[A_n \ldots A_j]}\sigma^j(A_j)_T$ are nested, closed, and have diameters going to 0 as j goes to $-\infty$. So there is a single point in the intersection of these sets.

(2-5) follow from the definitions. Note that (3) is a statement about points in $\mathbb{R}^d$, and (5) is a statement about configurations in $\mathbb{R}^d$. We point out (4) to help keep in mind both the meaning of λ and the meaning of $g_{[A_n \ldots A_0]}$. ($g_{[A_n \ldots A_0]}$ gives the position of $(A_0)_T$ in $(\sigma')^n(A_n)$.) □

We now define a pair of relations: $\approx$ is an equivalence relation on $\mathbb{A}^n$; $|$ is a reflexive, symmetric relation on blocks in $\mathbb{A}^n$:

For $\alpha, \beta \in \mathbb{A}^n$, $\alpha \approx \beta$ if and only if $\lambda(\alpha) = \lambda(\beta)$.

For blocks $A, B \subset \mathbb{A}^n$, $A|B$ if and only if there exist $\alpha \in A$, $\beta \in B$ with $\alpha \approx \beta$. Note that for blocks $A|B$ if and only if $\lambda(A) \subset \lambda(B)$ or $\lambda(B) \subset \lambda(A)$ or $\lambda(A) \cap \lambda(B) \neq \emptyset =$ int $\lambda(A) \subset$ int $\lambda(B)$ (This follows from our condition on T, that all the elements of $\{w^n A \mid A \in T, n \in \mathbb{Z}\}$ be disjoint.) could define for $\cup\mathbb{A}^n$.

In an abuse of notation we define $|$ on supertiles of each level, n as $g(\sigma')^n(A)|h(\sigma')^n(B)$, $A, B \in S$, $g, h \in G$ if and only if there exists a $k \in N$, $k > n$, $[\alpha_k \ldots], [\beta_k \ldots] \in \mathbb{A}^k$ with $[\alpha_k \ldots]|[\beta_k \ldots]$, $g = g_{[\alpha_k \ldots \alpha_{n+1}]}, h = g_{[\beta_k \ldots \beta_{n+1}]}$, $A = \alpha_n, B = \beta_n$ and finally $(\sigma')^n(A) = \{\lambda([\alpha_n \ldots \alpha_0])\}$, $(\sigma')^n(B) = \{\lambda([\beta_n \ldots \beta_0])\}$.

This can be interpreted as:

Lemma 1.2 *Let* $A, B \in S$, $g, h \in G$, $n \in N$. *Then* $g(\sigma')^n(A)|h(\sigma')^n(B)$ *if and only there exists some* $k \in \mathbb{N}$, $k > n$, $X \in T$ *with* $g^{(n-k-1)}A, h^{(n-k-1)}$ *adjacent tiles in* $(\sigma')^{k+1-n}(X)$.

This is not hard to verify and proof is omitted.

The definition of $\lambda([A_n \ldots])$ was "top-down". In particular $\lambda([A_n \ldots A_0])$ gives the position of a tile within a fixed supertile $(\sigma')^n((A_n)_T)$. To define a map from $\mathbb{A}$ we will need a "bottom-up" construction that fixes the position of a configuration given the position of a tile. To do this we define a host of maps:

For each $\alpha \in \mathbb{A}$, define

$\lambda_\alpha : [\alpha] \to \mathbb{R}^d$ as follows. Given $\beta \in [\alpha]$, let n be any integer such that for all $k \geq n$, $\alpha_k = \beta_k$.

$$\lambda_\alpha(\beta) := (g_{[A_n \ldots]})^{-1}\lambda([\beta_n \ldots])$$

Note that this does not depend on *which* n we use, and that $\lambda_\alpha(\alpha) = \mathsf{o}$. One ought verify that $g_{[A_n \ldots]} := \lim_{j \to -\infty}(g_{[A_n \ldots A_j]})$ is a well-defined affine transformation in G.

We can now define **infinite level supertiles** as configurations of the form $\tau_\alpha := \{\lambda_\alpha[\ldots \beta_0] \mid \beta \in [\alpha]\}$, and of the form $g\tau_\alpha$, $g \in G$.

Note for any infinite level supertile $g\tau_\beta$, that $\mathsf{o} \in g\tau_\beta$ if and only if there is a $\alpha \in [\beta]$ such that $g\tau_\beta = \tau_\alpha$.

We define $|$ on infinite-to-the-left-blocks as follows. Let $\alpha, \beta \in \mathbb{A}$; then $[\alpha]|[\beta]$ if and only if there exist: $\alpha' \in [\alpha], \beta' \in [\beta]$, such that for all $m \in \mathbb{Z}$, there is an $n \in Z$, $n > m$, and $\gamma \in \mathbb{A}$ such that $\alpha'_{m-1}, \beta'_{m-1} \in (\gamma_m)^+$ and $[\gamma_n \dots \gamma_m \alpha'_{m-1}]|[\gamma_n \dots \gamma_m \beta'_{m-1}]$.

This can be interpreted as follows (we skip the proof):

Lemma 1.3 *If $\tau \in \Sigma(T, \sigma')$ contains infinite-level supertiles $g\tau_\alpha$, $h\tau_\beta$ such that $\llbracket g\tau_\alpha \rrbracket \cap \llbracket g\tau_\beta \rrbracket \neq \emptyset = \text{int } \llbracket g\tau_\alpha \rrbracket \cap \text{int } \llbracket g\tau_\beta \rrbracket$, then $[\alpha]|[\beta]$*

Accordingly, in an abuse of notation we define $|$ on infinite level supertiles as $g\tau_\alpha|h\tau_\beta$ if and only if $[\alpha]|[\beta]$ or $\llbracket g\tau_\alpha \rrbracket \cap \llbracket h\tau_\beta \rrbracket = \emptyset$.

2. Examples

We would be well served by examples before continuing.

Example 2.1

Our first example is the L-tiling or "chair" tiling in $\mathbb{R}^2$. T consists of one tile, L; σ doubles all distances; σ' expands each L-tile and replaces each σL with four L-tiles. L, $\sigma' L$, and $(\sigma')^2 L$ are shown in figure 1(i). In figure 1(ii) is a portion of what appears to be an element of $\Sigma(\{L\}, \sigma')$.

In figure 1(iii) we have illustrated $S = L^+ = \{a, b, c, d\}$. $\mathbb{A}$ consists of all bi-infinite strings of these letters; $\Gamma(\{L\}, \sigma')$ is shown in (iv).

In figure 2 we have illustrated some things connected to our first map λ. Recall that S can be regarded as both a set of tiles and as a set of letters. The elements of S are in boldface when used to indicate tiles.

In figure 3 we have illustrated some infinite level supertiles. Note the location of o!

It can be noted that the L-substitution species has unique decomposition— every L-substitution tiling arises in exactly one way as the image of another L-substitution tiling under the map σ'. (That is, one can uniquely cluster together the L-tiles into 1-level supertiles.

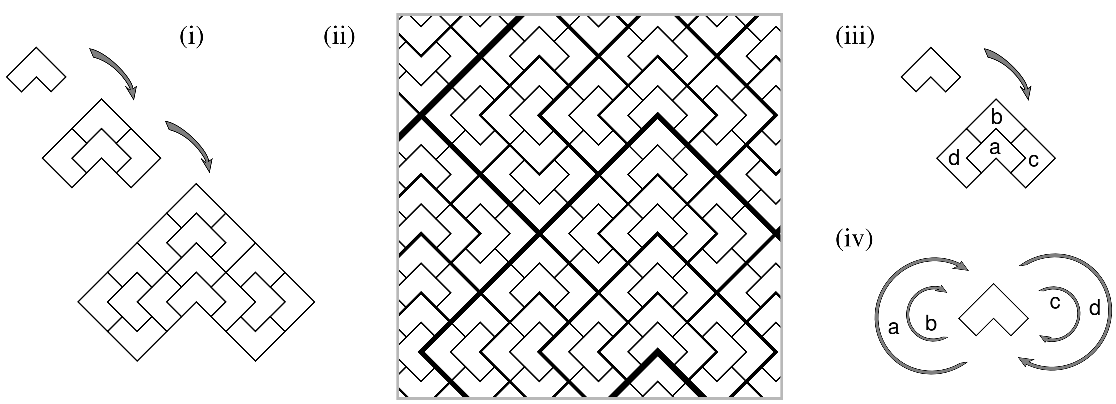

Figure 1: The L substitution tiling

Example 2.2 Our second example is quite familiar; T consists of one tile , a segment in $\mathbb{R}^1$. σ' simply doubles the length of the segment and replaces it with two segments. S will be denoted

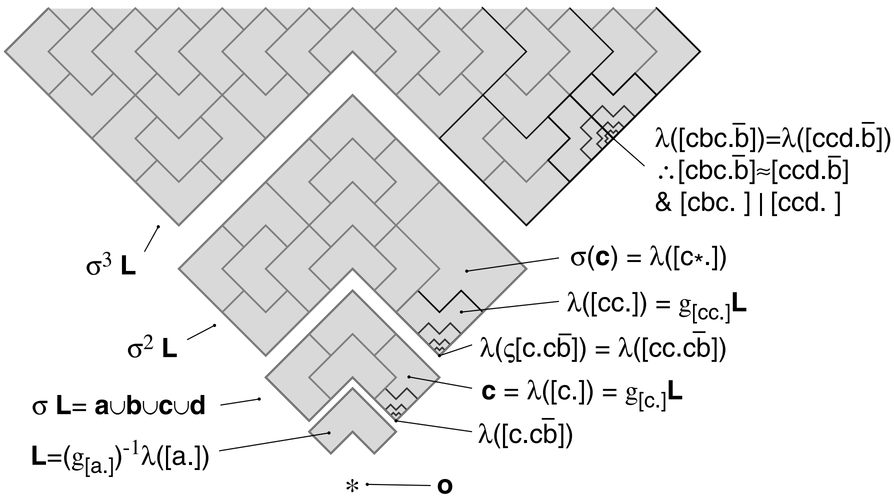

Figure 2: λ and the L substitution tiling

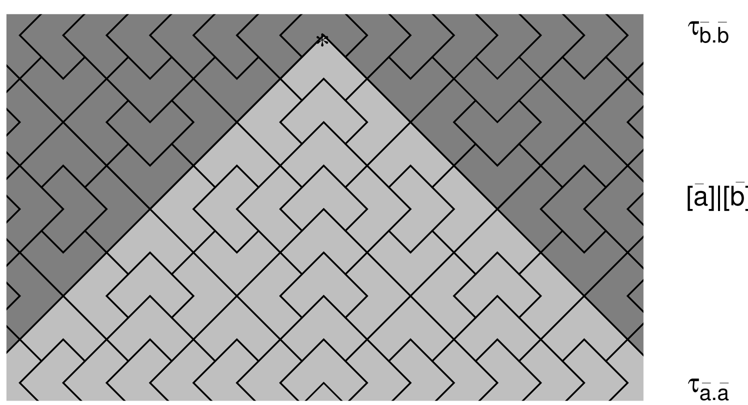

Figure 3: Two infinite level supertiles in the L substitution tiling

$\{0,1\}$ and $\mathbb{A}$ consists of all bi-infinite strings of these digits. One special infinite supertile, $\tau_{\overline{0}}$ can be regarded as the binary representation of the non-negative real numbers. Note for example that $\overline{0}100.1\overline{1} \approx \overline{0}101.0\overline{0}$ and that $[100.]|[101]$.

This example does not have unique decomposition— all tilings in the species are equivalent up to translation and σ' is two-to-one.

Example 2.3 Here T consists of the three triangles shown; σ is a dilation of magnitude $s \approx 1.324717957244746$, the real root of $s^3 - s - 1 = 0$. σ' takes the elements of T to supertiles as shown. A small portion of a substitution tiling in $\Sigma(T, \sigma')$ is at right in (iii).

S and the substitution graph are illustrated in figure 4(ii).

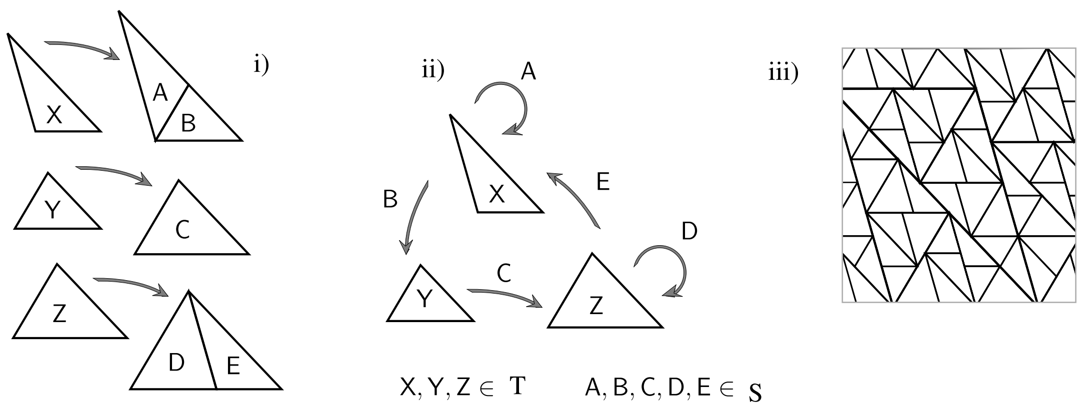

Figure 4: A substitution on triangles

3. Beginning Results

For the following, fix G, T and σ and σ' in $\mathbb{R}^d$. Then S and λ are determined.

Theorem 3.1 *1. Infinite level supertiles are well defined, unbounded configurations in $C(T)$.*

2. For $\alpha, \beta \in \mathbb{A}$, if $\beta \in [\alpha]$ then there is a $g \in G$ with $g\tau_\alpha = \tau_\beta$. If $\Sigma(T, \sigma')$ has unique-decomposition then the converse holds as well. Also, $\tau_{\varsigma\alpha} = \sigma'\tau_\alpha$.

This essentially follows from the definitions and is not hard to verify. Note that in fact, we can explicitly give the g in (2) above: if $\beta \in [\alpha]$, there exists an $n \in \mathbb{N}$ such that for all $k > n$, $\alpha_k = \beta_k$. Then for $g = g_{[\beta_n \dots]}(g_{[\alpha_n \dots]})^{-1}$, we have $g\tau_\alpha = \tau_\beta$.

The second and third parts of the following are well known "folk-theorems". The statements strongly depend on the tiles in T having non-empty interior.

Theorem 3.2 *1. There is an infinite level supertile that is a tiling in $\Sigma(T, \sigma')$.*

2. Thus, $\Sigma(T, \sigma')$ is non-empty.

3. In fact, if $\Sigma(T, \sigma')$ has unique decomposition, then $\Sigma(T, \sigma')$ has uncountably many non-congruent tilings.

Proof (1) Since $\sigma(A)$ has greater measure than A, for each element of T, it follows that $|S| > |T|$. We claim that there is an $\alpha \in \mathbb{A}$ satisfying:
(i) There is an $n \in \mathbb{N}$ with $\varsigma^n \alpha = \alpha$ and
(ii) $\lambda(\alpha_0.\alpha_{-1}\ldots) \in int\lambda([\alpha_0])$

We construct the desired α as follows: First note that there must be some $A \in S$ with $gA \in int((\sigma')^n(A))$ for some $g \in G, n \in \mathbb{N}$. This must be so since σ expands all distances and each tile is bounded; thus the number of tiles strictly in the interior of each $(\sigma')^n(A)$, $A \in T$ must grow without bound as n grows. Since T is finite, we must have that there exists some $A \in T$, $g \in G$, $n \in \mathbb{N}$ with $gA \subset int(\sigma')^n A$.

Now this $g = g_{[A_n..A_0]}$ for some block $[A_n..A_0] \subset \mathbb{A}$ with $A_n = A_0 = A$. Take $\alpha \in \mathbb{A}$ to be such that $\alpha_i = A_k$ where $i \equiv k \text{mod } (n-1)$. Then this α satisfies our claim

Now it is not difficult to see that $\lambda_\alpha([\alpha]) \in \Sigma(T,\sigma')$— in other words, $[\![\lambda_\alpha([\alpha])]\!] = \mathbb{R}^d$.

(2) Thus $\Sigma(T,\sigma')$ is not empty.

(3) Now we show that $\Sigma(T,\sigma')$ is uncountable if $\Sigma(T,\sigma')$ has unique decomposition. From the proof of 1 we can quickly see that there is an $A \in S$, $n \in N$ such that there are two distinct images of A in the interior of $(\sigma')^n(A)$. Let us denote these images $B = \lambda([B_n \ldots B_0].)$ and $C = \lambda([C_n \ldots C_0].)$, $B_n = B_0 = C_n = C_0 = A$. Now define $f : \{0,1\}^{\mathbb{Z}} \to \Sigma(T,\sigma')$ by:

$f(s) = \tau_\alpha$ where, for $j,k \in \mathbb{Z}, 0 \le k < n$, $\alpha_{jn+k} = B_k$ if $s_j = 0$, $\alpha_{jn+k} = C_k$ if $s_j = 1$. As before, it is not hard to verify that each such τ_α has support $[\![\tau_\alpha]\!] = \mathbb{R}^d$ and thus is a tiling in $\Sigma(T,\sigma')$.

We next claim that if $\Sigma(T,\sigma')$ has unique decomposition, none of the images of f can be congruent to one another. This follows from Theorem 3.1,(2) since for all $\alpha,\beta \in f(\{0,1\}^{\mathbb{Z}})$, $\alpha \notin [\beta]$. Moreover f is one-to-one, and so 3 is proved. ☐

The following theorems show that these definitions really capture the structure of $\Sigma(T,\sigma')$.

Proposition 3.3 *The elements of $\Sigma(T,\sigma')$ are precisely of the form $\bigcup g_i\tau_{\alpha_i}$ where $\{\alpha_i\}$ is a countable subset of $\mathbb{A}$ with $g_i\tau_{\alpha_i}|g_j\tau_{\alpha_j}$ for each distinct pair i,j.*

We would like to go on to add that in fact $\cap[\![g_i\tau_\alpha]\!]$ consists of a single point, and that there can be only finitely many such $g_i\tau_{\alpha_i}$; we can't see a proof though, and perhaps this is not true for all $\Sigma(T,\sigma')$.

Proof Before we begin, we show that for any point $x \in \mathbb{R}^d$, for any $n \in \mathbb{N}$, there is a supertile $g(\sigma')^n(A) \subset \tau$ with $x \in \sigma^n(A)$. To see this, let d be the maximum diameter of any tile in T, and let D be a ball of radius $2d$ with center $w^{-n}(x)$. Then consider the configuration τ' of tiles in τ with support intersecting $w^n D$. Then (by the definition of $\Sigma(T,\sigma')$) there is a supertile $g(\sigma')^k(B) \supset \tau'$. Now we must have $k > n$ and that $g(\sigma')^k(B)$ is the disjoint union of supertiles of the form $\lambda([\beta_k \ldots \beta_n])$ with $\beta_k = B$. One of these must have support containing x, and is therefore contained in τ' and hence in τ.

We now show that for all $x \in \mathbb{R}^d$ there is an infinite level supertile $g\tau_\alpha \subset \tau$ with $x \in [\![g\tau_\alpha]\!]$. It suffices to show that any $\tau \in \Sigma(T,\sigma')$ there is an infinite-level supertile $\tau_\alpha \subset \tau$ with $\mathsf{o} \in [\![\tau_\alpha]\!]$.

Let S_n be the collection of all n-level supertiles in τ with support containing o. Then if $h(w')^n(A) \in S_n$, there must be a $B \in A^+$ with $hg_A^{(n)}(w')^{n-1}(B) \in S_n$. Now since no S_n is empty, there must be, in fact, a sequence of A_i, $i = 0,1,\ldots$ such that there is a $h_i \in G$ with $h_i(w')^n(A_i) \in S_n$, and $A_{i-1} \in A_i^+$ with $h_i g_{A_i}^{(n)} = h_{i-1}$. Now there exists an $[B_0 \ldots] \in \mathbb{A}^0]$ such

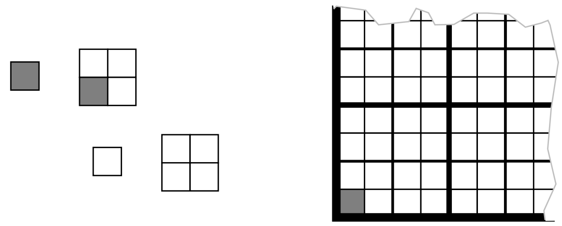

Figure 5: Outside of the domain of Theorem 3.5

that $A_0 = B_0$ and $(g_{[B_0 \ldots]})^{-1}(B_0)_T = A_0$. Then it is not hard to verify that for $\alpha \in \mathbb{A}$ with $\alpha = \ldots A_n \ldots A_0.B_{-1} \ldots$, $\tau_\alpha \subset \tau$ with $\mathsf{o} \in [\![\tau_\alpha]\!]$.

We now claim there is a collection of infinite-level supertiles in τ with support $\mathbb{R}^d$. We first define a collection $\{g_i \tau_{\alpha_i}\}_{1 \le i \le n}$ of infinite level-supertiles to be **o**k if and only if each $g_i \tau_{\alpha_i} \subset \tau$ and for all distinct i, j, $g_i \tau_{\alpha_i} | g_j \tau_{\alpha_j}$.

We prove the following statement by induction on n: If $\{g_i \tau_{\alpha_i}\}_{1 \le i \le n}$ is a finite o.k. collection of infinite-level supertiles then either $\tau = \bigcup g_i \tau_{\alpha_i}$ or $\{g_{n+1} \tau_{\alpha_{n+1}}\}$ such that $\{g_i \tau_{\alpha_i}\}_{1 \le i \le n+1}$ is o.k.

If $\tau = \bigcup g_i \tau_{\alpha_i}$ then we are done; otherwise there is an $x \in \mathbb{R}^d$ not in the interior of the support of any of the $g_i \tau_{\alpha_i}$. With no loss of generality, we may assume $x = \mathsf{o}$ and that o is on the boundary of one or more of the $g_i \tau_{\alpha_i}$. Essentially by repeating the argument at the beginning of the proof we can construct the needed $g_{n+1} \tau_{\alpha_{n+1}}$ such that $\{g_i \tau_{\alpha_i}\}_{i \le n+1}$ is o.k. $\square$

The following is a well-known "folk theorem". In a way, this idea is already present in [ ] (even though he is not, strictly speaking, considering substitution tilings). We proved this within the proof of Proposition 3.3.

Theorem 3.4 *For any $\tau \in \Sigma(T, \sigma')$, for any tile A in τ:*

1. *for any $n \in \mathbb{N}$, there is a $g \in G$, $B \in T$ such that the n-level supertile $g(\sigma')^n(B)$ is a subset of τ and $A \in g(\sigma')^n(B)$.*
2. *There is an $g \in G$, $\alpha \in \mathbb{A}$ such that the infinite level supertile $g\tau_\alpha$ contains A.*

This provides a kind of converse to Proposition 3.3.

Theorem 3.5 *If* **S** *is irreducible, then every infinite level supertile is a subset of some element of $\Sigma(T, \sigma')$. In fact, if $g \ldots$ ****

It is not hard to cook up examples in which mS is not irreducible and there is an infinite level supertile that does not appear in any element of $\Sigma(T, \sigma')$. Consider the following example: M is R^2, G consists only of translations. There are two tiles in T, colored squares, with the substitutions shown. The infinite-level supertile pictured at right is not in any tiling in $\Sigma(T, \sigma')$.
Proof of Theorem 3.5: } One can verify that if **S** is irreducible, then for every pair $A, B \in S$ there exists a $g \in G$, $n \in \mathbb{N}$ with $gA \in (w')^n(B)$, $gA \subset int(w^n(B))$.

Now let τ_α, $\alpha \in \mathbb{A}$ be any infinite level supertile. If $[\![tau_\alpha]\!] = \mathbb{R}^d$ then $\tau_\alpha \in \Sigma(T, \sigma')$ and we are done. So assume $[\![\tau_\alpha]\!] \neq \mathbb{R}^d$.

Choose any tile $h'A$, $h' \in G, A \in T$ on the boundary of τ_α. τ_α. We may assume, after possibly applying an isometry g and relabeling α, that h is the trivial isometry (the identity of G), and that $\alpha_0 = A$. Check out addresses of points in τ_α

Let D_0 be a ball centered at o with radius $2d$, where d is the maximum diameter of the tiles in T, and let $D_k = \sigma^k D_0$.

Let Σ_k consist of all configurations τ' in $C(T)$ such that

(i) $[\![\tau']\!] \supset D_k$;

(ii) there exists a supertile $g(\sigma')^n(B) \supset \tau_k$; and

(iii) $\tau_k \supset \{A \in \tau_\alpha \mid [\![A]\!] \cap D_n \neq \emptyset\}$.

We will show that there exists a sequence of $\tau_k \in \Sigma_k$, with $\tau_k \subset \tau_{k+1}$. The union of the τ_k in this sequence will be a well-defined tiling in $\Sigma(T, \sigma')$, containing τ_α.

Claim: each Σ_k is non-empty

Proof of claim: ***By the definition of τ_α, there must exist an n such that $\lambda'_h(A_n \dots A_0) \supset D_k \cap \lambda'_h(\dots A_0)$. Now there exists an $m \in \mathbb{N}$, $h'' \in G$, $B \in T$ such that $h''(A_n)_T$ is in the interior of $(\sigma')^m(B)$.

Note that $(h'')^{(n)}(\sigma')^n(A_n) \subset (\sigma')^{m+n}$ and that for H ($(h'')^{(n)}(\sigma')^n(A_n) = \lambda'_h(A_{n \dots A_0})$,

$$H = h\ (g_{A_n \dots A_1})^{-1}\ ((h'')^{(n)})^{-1}$$

Now choosing m suitably large, we can assume that

$$H^{-1}\ D_k \subset \sigma^{m+n}(B)$$

Define then $H^{-1}\Sigma_k$ to be $(h'')^{(n)}(\sigma')^n(A_n) \cup ((\sigma')^{m+n} \cap H^{-1})\ D_k$.

The claim is proved.

Note that by the definition of each Σ_k, if $\tau \in \Sigma^k$, $\tau \cap D_{k-1} \in \Sigma_{k-1}$; thus there is a sequence $\{\tau_k \in \Sigma_k\}$, $\tau_k \subset \tau_{k+1}$, and $\cup\tau_k$ is a tiling in $\Sigma(T, \sigma')$ containing τ_α. □

4. Algorithmic descriptions of orbits in $\Sigma(T, \sigma')$ under σ'

Theorem 4.1 *$\sigma' : \Sigma(T, \sigma') \to \Sigma(T, \sigma')$ is onto.*

Proposition 4.2 *If there are finitely many local configurations in $\Sigma(T, \sigma')$, then there is a Mealy machine that precisely determines the relation $\mid$ on $\mathbb{A}$.*

Corollary 4.3 *There is an algorithm for determing the finite orbits of σ' acting on $\Sigma(T, \sigma')$. If $\Sigma(T, \sigma')$ has finitely many local configurations there is an algorithm for determining all orbits of σ' acting on $\Sigma(T, \sigma')$.*

Corollary 4.4 *Given $\Sigma(T, \sigma')$ there are countably many self-similar tilings in $\Sigma(T, \sigma')$, finitely many of each inflation factor.*

The following is the result of a calculation based on Corollary 4.3. Of course, infinitely many similar sounding corollaries are also available!

Corollary 4.5 *Under the action of σ', the L substitution species has precisely four orbits of length 1, six orbits of length 2 and sixteen orbits of length three.*

Theorem 4.6 *If $\Sigma(T, \sigma')$,$\Sigma(T_0, \sigma'_0)$ are well enough behaved (polyhedral tiles, her. facets and sibling v2v) and have homomorphic addressings, then in some fundamental they are the same tiling. ...*