Regular Production Systems and Tilings in the Hyperbolic Plane

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In any given fixed “nice” setting— e.g. tilings by polygonal tiles in the hyperbolic plane— we can ask several closely related questions:

Is the Domino Problem decidable?

That is, is there an algorithm to decide whether any given set of tiles admits a tiling?

Does there exist an “aperiodic” set of tiles? Or more precisely:

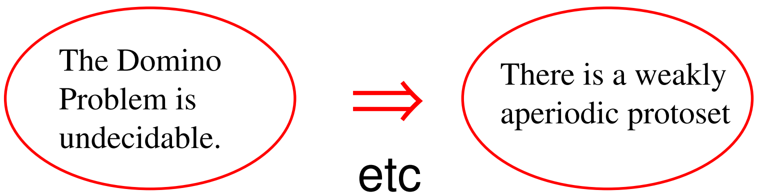
Is there a weakly aperiodic set of tiles?

Is there a set of tiles that does admit a tiling, but admits no tiling with a compact fundamental domain?

Is there a strongly aperiodic set of tiles?

Is there a set of tiles that does admit a tiling, but admits no tiling with an infinite cyclic symmetry?

Wang (1961) pointed out that:

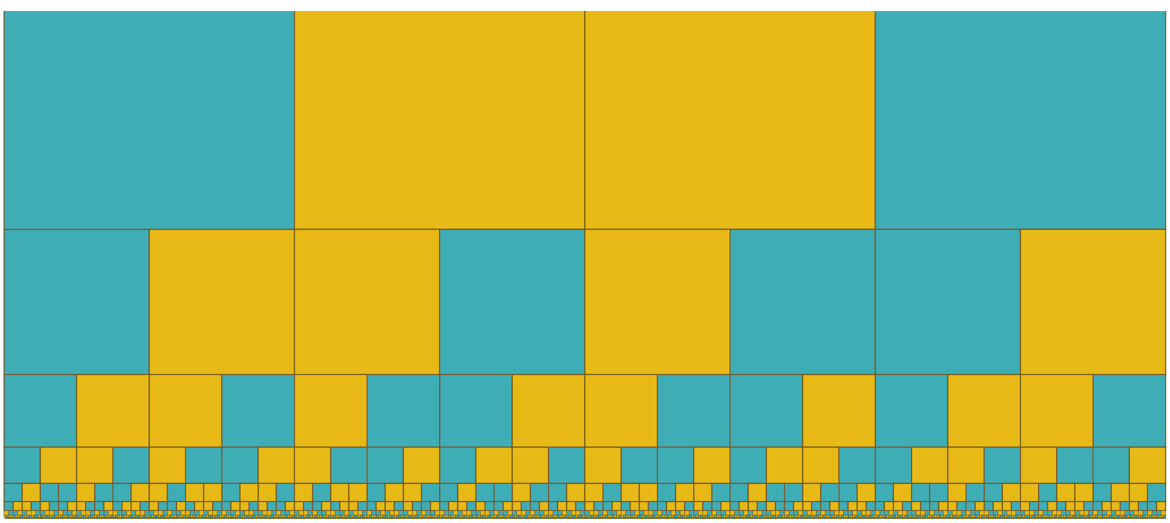


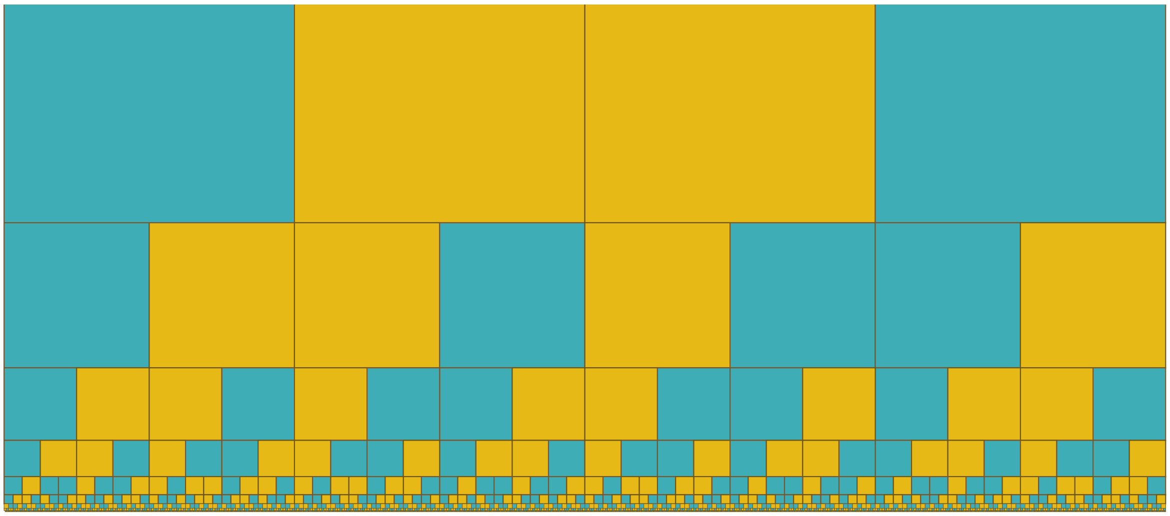
Berger (1966) essentially showed the Domino Problem is undecidable for polyhedral tiles in E^n.

In the hyperbolic plane, examples of weakly aperiodic protosets have been known for over twenty years. Other than the undecidablity of the completion problem, no other of these questions had been solved in $\mathbf{H}^2$.

I'd like to discuss a model of tiling in the hyperbolic plane that, at least, allows the solution of a wide range of interesting problems, including the construction of the first "strongly aperiodic" set of tiles in $\mathbf{H}^2$.

Here is a fairly typical example of a weakly aperiodic protoset in $\mathbf{H}^2$, based on the Morse sustitution system $0 \to 01$, $1 \to 10$; the horocyclic rows in the tiling correspond to bi-infinite words in the system. A row is above another exactly when the corresponding words are related by substitution.





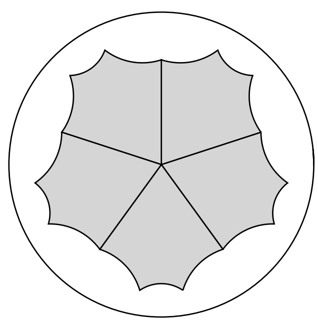
Though this protoset does not admit a tiling with compact fundamental domain, it does admit a tiling invariant under an infinite cyclic symmetry (and so is only weakly periodic). The periodic tilings precisely correspond to the periodic orbits under the substitution.

Incidentally, every primitive substitution system corresponds to a weakly aperiodic set of tiles in H^2 in precisely this manner.

This basic idea is quite general; let's consider a new proof of an old theorem:

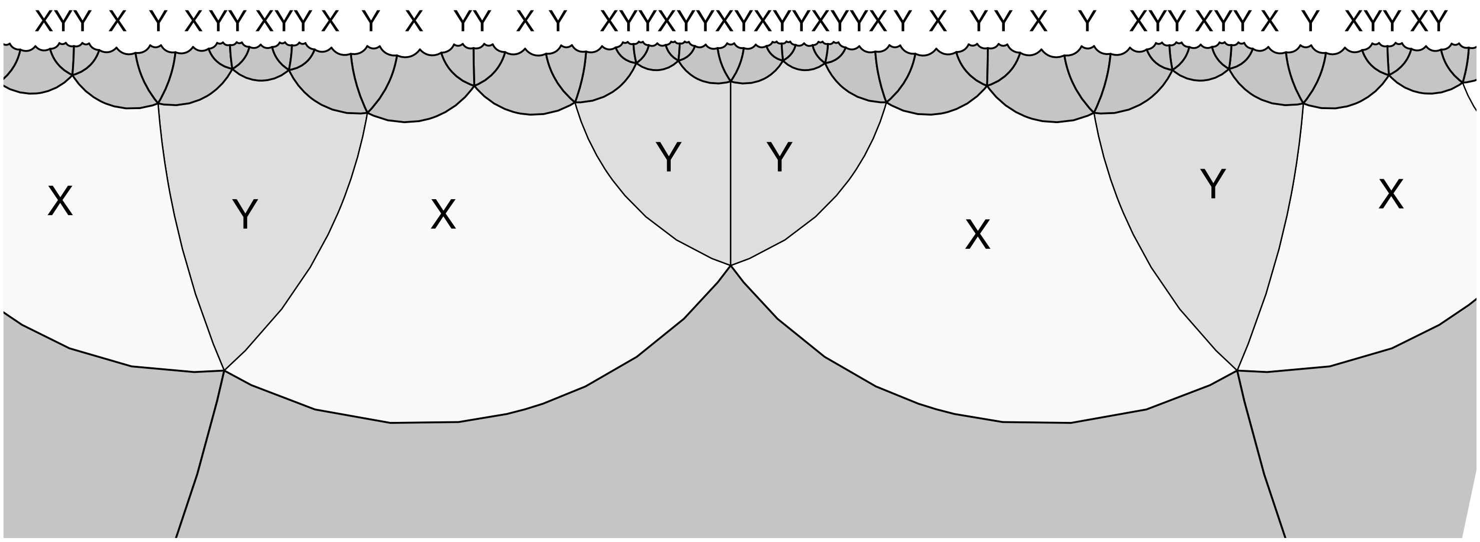
Theorem 1 *(Poincaré) For any* $p, q \geq 3$ *with* $\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$ *there is a tiling of* $\mathbf{H}^2$ *by regular* p*-gons meeting* q*-to-a-vertex.*

Proof We'll assume $p, q \geq 4$; the other cases can be handled in very similar fashion.



Now at least locally, this is more or less trivial. With simple trigonometry, one can construct a regular p-gon with vertex angles $2\pi/q$; and thus construct an arrangement of q regular p-gons surrounding a single vertex. The question is, how to extend this to a global tiling.

The basic idea is that letters will represent possible configurations of tiles along a band. Words in these letters represent bands.



Not every sequence of letters makes combinatorial sense, so we restrict ourselves to a certain language— a set of words— that at least locally are correct.

A relation describes which bands may fit together. This relation is quite like a substitution system.

Finally, more or less, orbits describe tilings.

Consider the alphabet $A = \{X, Y\}$ and define a regular language

$$L := (((XY^{q-3})^{p-4} + (XY^{q-3})^{p-3})XY^{q-4})^*$$

That is, we can concatonate as many strings as we please of the form: either $p-3$ or $p-4$ strings of XY^{q-3}, followed by XY^{q-4}.

For example, if we take p=5, q=5,

$$L = ((XYY + XYY\ XYY)XY)^*$$

and a typical word is

$$XYY\ XY\ XYY\ XY\ XYY\ XYY\ XY$$

Now consider the free language $A^* = (X+Y)^*$, arbitrary strings of X's and Y's. Consider a substitution system: a map s defined on A^* by

$$X \to (XY^{q-3})^{p-4} XY^{q-4}$$

$$Y \to (XY^{q-3})^{p-3} XY^{q-4}$$

For example, when $p = q = 5$, we have

$$X \to XYY\ XY$$

and

$$Y \to XYY\ XYY\ XY$$

Thus

$$XYX \to XYYXY\ \ XYYXYYXY\ \ XYYXY$$

Now this map s extends to the bi-infinite strings $A^{\mathbf{Z}}$. Classically, there exists an orbit $\{w_i\} \subset A^{\mathbf{Z}}$. That is, there is a set of infinite strings such that $s(w_i) = w_{i+1}$ for all i. Indeed, there are uncountably many such orbits and countably many are periodic.

Now let $L^\infty \subset A^{\mathbf{Z}}$ the extension of L to infinite words. That is, L^∞ consists of the bi-infinite strings such that every finite substring is a substring of some element of L.

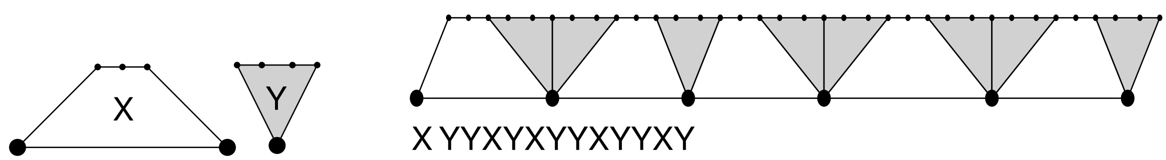
It is not difficult to prove that

(*) $s(L^\infty) \subset L^\infty$ and

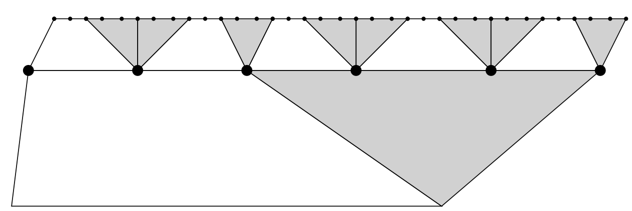
(*) for any orbit $\{w_i\}$in A^* under s, each $w_i \in L^\infty$.

Now for the geometric interpretation:

(1) Every word in L corresponds to an abstract combinatorial strip of p-gons meeting q-to-a-vertex. The words in L^∞ correspond to infinite strips.

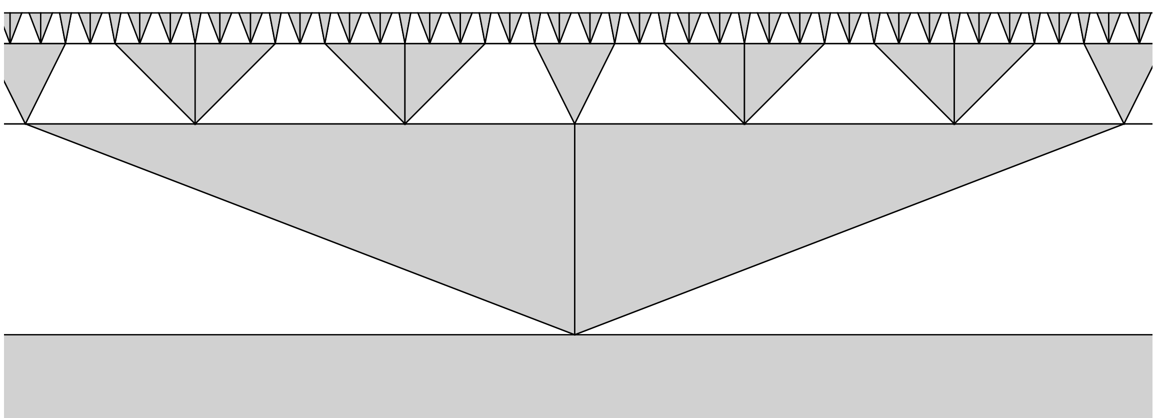


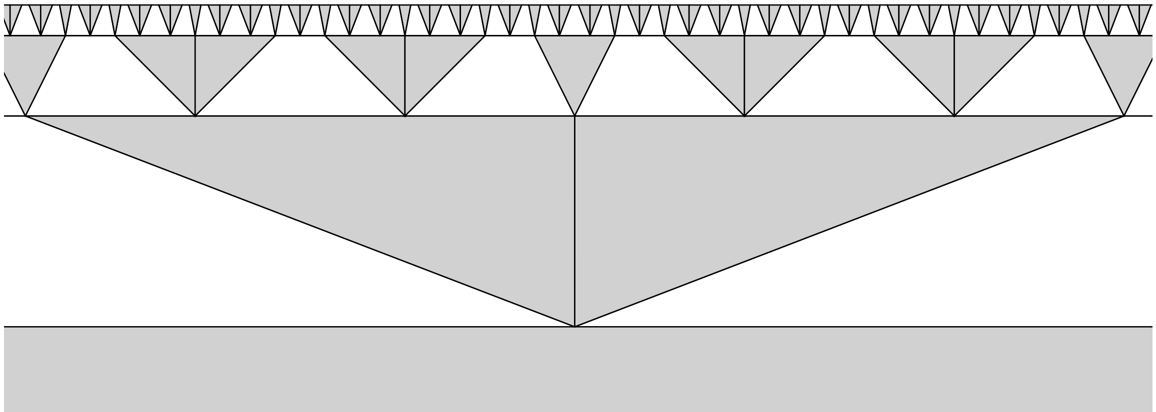
(2) If $s(w) = v$, the strip corresponding to w fits above the strip corresponding to v.



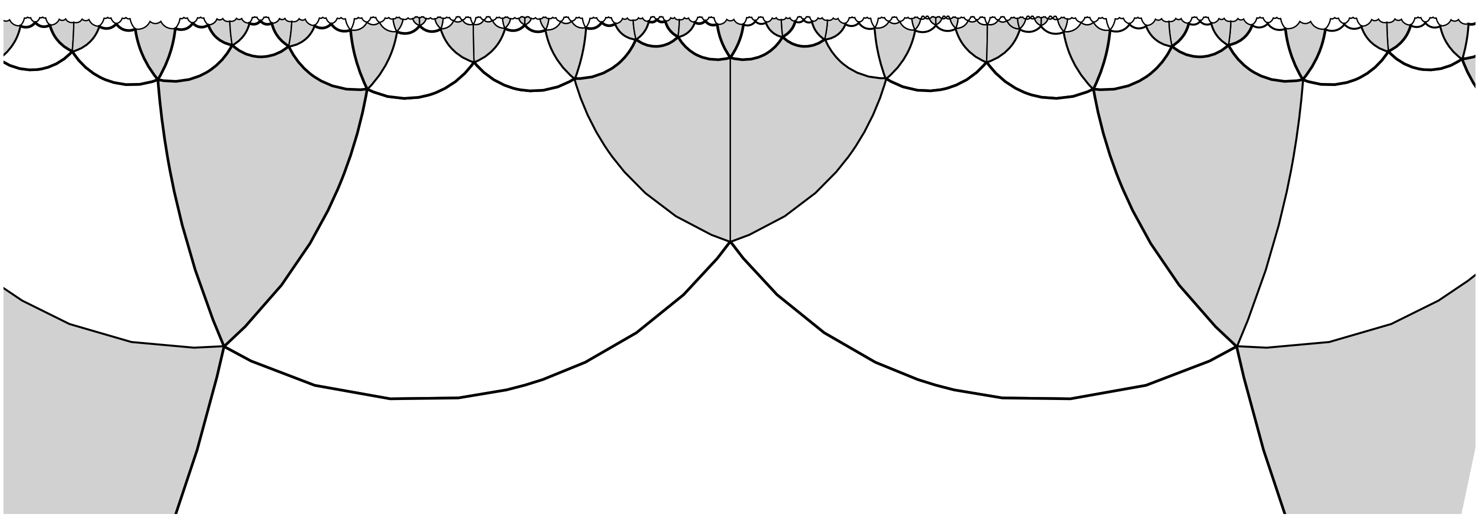
$$XY \mapsto XYY\ XY\ XYY\ XYY\ XY$$

(3) Consequently, any orbit corresponds to an abstract complex of p-gons meeting q-to-a-vertex. This complex is unbounded and simply connected.





Finally, for some magic: using the original geometric arrangement of q regular p-gons meeting at a vertex, we may chart a geometry on this complex. This geometry will be complete, with constant negative curvature. *VOILA!* Our complex *is* a tiled hyperbolic plane! □



The general idea is thus:

A set of tiles T is a set of local combinatorial constraints.

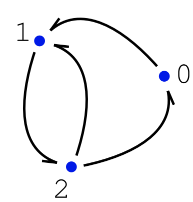
Relate a language, with a map, and orbits to abstract complexes with the combinatorial structure of tilings. These complexes exist iff there exist such orbits.

These complices in turn correspond to tilings, by charting the geometry of the original tiles and local arrangements.

Now in general this is much more subtle than our example. In the proof we just gave, we were able to find a classical symbolic substitution system within our construction. We thus were able to construct orbits, and indeed periodic orbits. But in general, there may be no orbits, or there may be orbits but no periodic orbits.

Consider the following example:

The language L is defined as paths in this graph:



Now take as rules

$$0 \mapsto 12$$

$$1 \mapsto 12 \qquad 1 \mapsto 21$$

$$2 \mapsto 01 \qquad 2 \mapsto 20$$

For words $w, v \in L$ write $w \mapsto v$ iff there is *some choice* of replacements of the letters in w yielding v. Note that a given word may map to no, one, or several other words.

For example:

$012120 \mapsto$

$0120 \mapsto$

$1212 \mapsto$

As before this relation extends to a relation on the infinite strings L^∞. *Is there an orbit?*

A “regular production system” (A, L, R) is specified by:

a regular language L on an alphabet A; and a relation R, satisfying certain axioms, on words in this language. The relation extends to a relation on L^{∞}.

In general, ask, given (A, L, R) are there orbits in L^{∞}?

Are there periodic orbits?

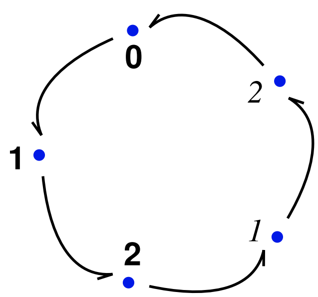
Quite unlike the classical symbolic substitution dynamical case, this is quite subtle. Indeed, there is an example of an (A, L, R) for which there *is* an orbit, yet there is *no* periodic orbit.

(This example can also be interpreted as a strongly aperiodic set of tiles in $\mathbf{H}^2$).

It seems quite likely that in general it is undecidable, given a regular production system, whether there exist orbits at all.

In many special situations, however, an orbit can be produced, by embedding a traditional symbolic substitution system within the regular production system.

For example, define the alphabet and language L' given by this graph, and take the obvious map $\phi : L' \to L$.



Now take the rules

$$0 \mapsto \mathbf{12}$$
$$\mathbf{1} \mapsto 12 \qquad 1 \mapsto \mathbf{21}$$
$$\mathbf{2} \mapsto 01 \qquad 2 \mapsto 20$$

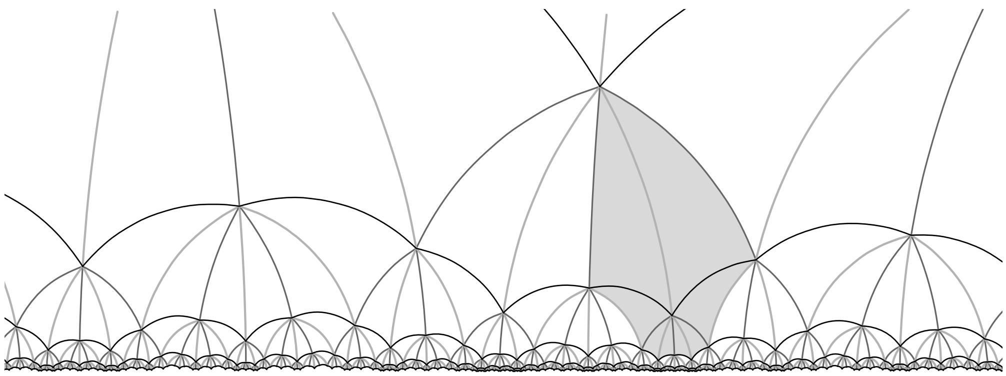
Note that for any $w, v \in L'$, if $w \mapsto v$, then $\phi(w) \to \phi(v)$. Now there is an action $\mapsto$ on $(A')^{\mathbf{Z}}$; classically, there are periodic orbits within $(A')^{\mathbf{Z}}$ under this action. It is not hard to show that in fact every orbit in fact lies within $(L')^{\infty}$ and so is mapped by ϕ into a periodic orbit within L^{∞}.

But again this sort of trick can't always work, since there are examples with orbits, but no periodic orbits.

These gadgets have proven to be quite powerful for analyzing combinatorial problems in H^2, and seem to be interesting in their own right.

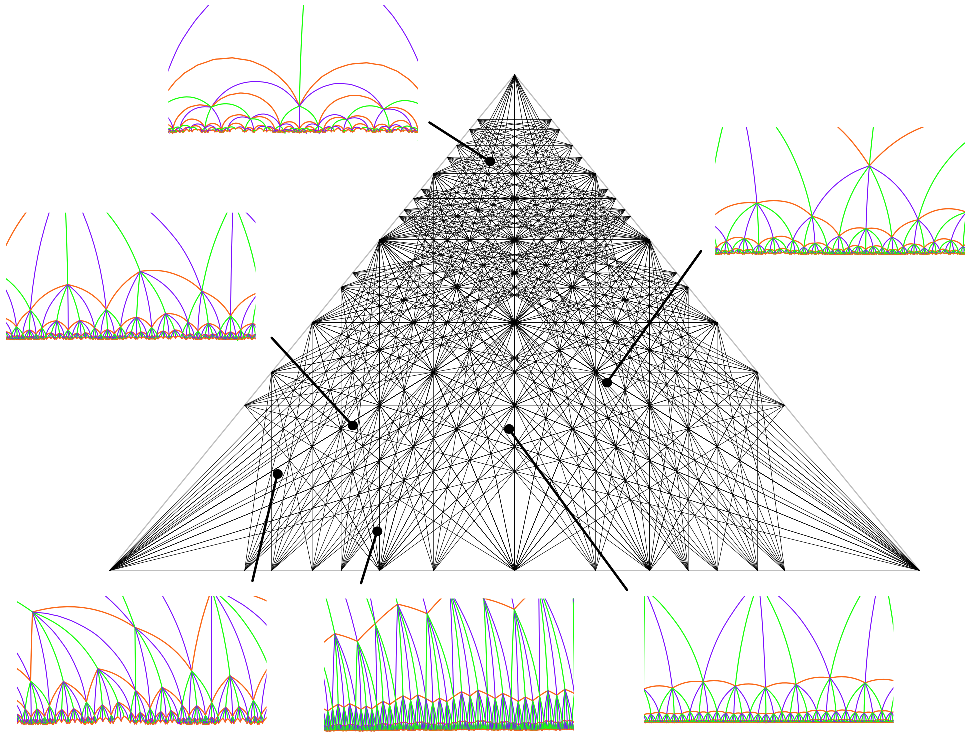
The strategy is: given a set of tiles, encode the combinatorics through a regular production system. If there exists an orbit in the system, then there exists a connected, simply connected, complete complex with the combinatorics of a tiling by the original tiles. Charting the geometry, our complex is now endowed with constant curvature and so *is* a tiling by the desired tiles.

I'd like to conclude with a final application:



This is a tiling by a triangle whose angles a, b, c satisfy $2a+2b+4c = 2\pi$. It admits no periodic tiling as the angles were chosen so that the area is not in $\mathbf{Q}\pi$. However, the triangle does admit a tiling with an infinite cyclic symmetry.

Any triangle whose angles satisfy the same equation will admit "homeomorphic" tilings.



Conjecture 2 *A triangle in* $\mathbf{H}^2, \mathbf{E}^2$ *does not admit a tiling if and only if it does not admit a "third corona". Every triangle that does tile admits a tiling with at least an infinite cyclic symmetry.*

In fact, the conjecture is now proven on all but a small set of triangles. The surprise is that a measure-1 set of triangles that do tile are in fact *weakly aperiodic*.

Here is a typical theorem:

Theorem 3 *Let T be a triangle in the hyperbolic plane with vertex angles a, b, c. Suppose there exist unique integers $r, s, t \geq 0$ with $ra + sb + tc = 2\pi$. Then T admits a tiling iff $r \equiv s \equiv t \mod 2$ and either $r, s, t \geq 2$ or $r, s, t = 1$*

This covers some pretty strange triangles!