A quick advertisement for the forthcoming compendium

The Symmetry of Things

with John H. Conway, Heidi Burgiel

(2007)
The Symmetry of Things
The Symmetry of Things
Puzzles

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We begin with a question:
We begin with a question:

Can this tile be used to form a tiling of the entire plane?

trivial example
We begin with a question:

Can this tile be used to form a tiling of the entire plane?

How can you tell?
We begin with a question:

Can this tile be used to form a tiling of the entire plane?

How can you tell?

certainly not
We begin with a question:

Can this tile be used to form a tiling of the entire plane?

How can you tell?

Myers (2003)
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Mann (2007)

Is there a general method to tell whether or not a given tile admits a tiling?
Many of these examples are very badly behaved.

This example (Myers 2003) has *isohedral number* 10, the current world record.
Many of these examples are very badly behaved.

That is, the tile can form periodic tilings, but the tiles fall into at least ten orbits; equivalently, the smallest possible fundamental domain has ten tiles!

An isohedral number 10 example
Many of these examples are very badly behaved.

This example (Mann 1999) doesn’t admit a tiling, but you can form quite large patches before things fall apart.
Many of these examples are very badly behaved.

It has *Heesch number* 5, the current world record— it can form a patch with five “coronas" and no more.
Many of these examples are very badly behaved.

There is no reason to suppose these are the worst possible examples—
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We have no idea (really) how they work, or what obstructions, if any, there are to creating increasingly terrible tiles.
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Again, *Is there a way to tell whether a given tile admits a tiling of the entire plane?*
The obvious algorithm to try is
See how far you can get!
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More precisely, we might enumerate all possible configurations by the tile, trying to cover larger and larger regions.
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• If the tile does not admit a tiling, then there will be some largest region that we cannot cover.

**Thm:** If we can cover arbitrarily large regions, we can cover the entire plane.
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*See how far you can get!*

More precisely, we might enumerate all possible configurations by the tile, trying to cover larger and larger regions.

- If the tile does *not* admit a tiling, then there will be some largest region that we cannot cover.

But this algorithm will never halt if the tile *does* admit a tiling.
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- If the tile does *not* admit a tiling, then there will be some largest region that we cannot cover.

But this algorithm will never halt if the tile *does* admit a tiling. So we make a modification.
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*See how far you can get!*

More precisely, we might enumerate all possible configurations by the tile, trying to cover larger and larger regions.

- If the tile does *not* admit a tiling, then there will be some largest region that we cannot cover.

- As we proceed, we check to see if any of our configurations could be a fundamental domain. If the tile admits a *periodic* tiling, we can discover this too.
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*See how far you can get!*

More precisely, we might enumerate all possible configurations by the tile, trying to cover larger and larger regions.

- If the tile does *not* admit a tiling, then there will be some largest region that we cannot cover.

- As we proceed, we check to see if any of our configurations could be a fundamental domain. If the tile admits a *periodic* tiling, we can discover this too.

This works fine— the algorithm will halt with a yes or no answer, so long as nothing falls through the gaps— so long as every tile either doesn’t admit a tiling or admits a periodic tiling.
Surely this is the case! ?
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Surely it is impossible for there to exist an *aperiodic* tile, that is, a tile that does admit a tiling of the plane, but never a periodic tiling.
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Surely it is impossible for there to exist an aperiodic tile, that is, a tile that does admit a tiling of the plane, but never a periodic tiling.

Such a tile would have to force some sort of bad behavior at all scales.
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Such a tile would have to force some sort of bad behavior at all scales.

And surely there is a general procedure (or theorem, or theory, or algorithm) that can determine whether a given monotile admits a tiling of the plane.
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How hard can this be?

The examples we see here should make us cautious.
The algorithm we outlined (enumerate all configurations, covering larger and larger disks, until we either crash, or find a fundamental domain) succeeds if there is no aperiodic tile.

If there is no aperiodic tile, then this algorithm solves both the Domino Problem and the Period Problem for any given monotile:

*The Domino Problem for Monotiles:*  
Given a tile, does it fail to admit a tiling?

*The Period Problem for Monotiles:*  
Given a tile, does it admit a periodic tiling?
That is:

The Period Problem is decidable for monotiles \iff There is no aperiodic monotile

The Domino Problem is decidable for monotiles
That is:

The Period Problem is undecidable for monotiles

The Domino Problem is undecidable for monotiles

There is an aperiodic monotile
Moreover, suppose there is a bound $H$ on Heesch number; i.e., suppose every tile that does not admit a tiling has Heesch number less than $H$. 
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Then we would have an algorithm for checking whether a given tile does not admit a tiling:
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Then we would have an algorithm for checking whether a given tile does not admit a tiling:

Simply enumerate larger and larger configurations, increasing the number of coronas (shells). If we find a configuration with more than $H$ coronas, we know the tile admits a tiling.
Similarly, suppose there is a bound $I$ on isohedral number; i.e. suppose that every tile that admits a periodic tiling has isohedral number less than $I$. 
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Then we would have an algorithm for checking whether a given tile admits a periodic tiling:

Simply enumerate all configurations with up to $I$ tiles; if we fail to find a fundamental domain, then the tile does not admit a periodic tiling.
And so we have

- Isohedral number is unbounded for monotiles
- Heesch number is unbounded for monotiles
- The Period Problem is undecidable for monotiles
- The Domino Problem is undecidable for monotiles
- There is an aperiodic monotile
And so we have

Isohedral number is unbounded for monotiles

The Period Problem is undecidable for monotiles

Heesch number is unbounded for monotiles

The Domino Problem is undecidable for monotiles

There is an aperiodic monotile

These are all open questions.
Myers, Mann and others are generating increasingly complex examples, via giant computer searches, that suggest that the Domino problem may well be undecidable for monotiles, and that there does exist an aperiodic monotile.
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But as with all such problems

(Is Question A decidable for combinatorial objects in X)

the searches are increasingly intractable, and the generic example does what it does for no particular reason.

(A “reason” is essentially an algorithm for deciding something!)

These tiles can be marketed as children’s toys, yet are an example of Undecidability in action!
These same questions can be asked in a range of settings:

Generally speaking, we specify a geometric space, and combinatorial restrictions on what we mean by “tile", “matching rule", etc.
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For example we might consider:

Tilings of $\mathbb{E}^2$, by a single tile (as we’ve just considered)
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For example we might consider:

(more specifically: Tilings of $\mathbb{E}^2$, by a single polygonal disk)
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Generally speaking, we specify a geometric space, and combinatorial restrictions on what we mean by “tile”, “matching rule”, etc.

For example we might consider:

Tilings of $\mathbb{E}^2$, by a set of tiles
These same questions can be asked in a range of settings:

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For example we might consider:

Tilings of $\mathbb{H}^2$ by . . .
These same questions can be asked in a range of settings:

Generally speaking, we specify a geometric space, and combinatorial restrictions on what we mean by “tile”, “matching rule”, etc.

For example we might consider:

Tilings of $\mathbb{E}^2$ by a shingle
These same questions can be asked in a range of settings:

Generally speaking, we specify a geometric space, and combinatorial restrictions on what we mean by “tile", “matching rule", etc.

For example we might consider:
Tilings of $\mathbb{E}^2$ by a disconnected tile

Each of these is strikingly different, and we must be explicit about which setting we are considering!
These same questions can be asked in a range of settings:

Generally speaking, we specify a geometric space, and combinatorial restrictions on what we mean by “tile”, “matching rule”, etc.

For example we might consider:

Tilings of $\mathbb{R}^2$, by tiles that have colored edges (colors must match)

Each of these is strikingly different, and we must be explicit about which setting we are considering!
These same questions can be asked in a range of settings:

Generally speaking, we specify a geometric space, and combinatorial restrictions on what we mean by “tile", “matching rule", etc.

For example we might consider:

Tilings of $\mathbb{E}^2$, by tiles that have colored edges, but satisfy an arbitrary relation (for example, red may meet with green or blue, but green may not meet blue)

Each of these is strikingly different, and we must be explicit about which setting we are considering!
These same questions can be asked in a range of settings:

Generally speaking, we specify a geometric space, and combinatorial restrictions on what we mean by “tile”, “matching rule”, etc.

For example we might consider:

Tilings of $\mathbb{E}^3$, by a single tile

Each of these is strikingly different, and we must be explicit about which setting we are considering!
These same questions can be asked in a range of settings:

Generally speaking, we specify a geometric space, and combinatorial restrictions on what we mean by “tile", “matching rule", etc.

For example we might consider:

Etc.

Each of these is strikingly different, and we must be explicit about which setting we are considering!
In any given specified setting, in which we can enumerate configurations we have:

- Isohedral number is unbounded
- Heesch number is unbounded
- The Period Problem is undecidable
- The Domino Problem is undecidable

There is an aperiodic set of tiles
In any given specified setting, in which we can enumerate configurations we have:

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(And there are more, similar kinds of implications)
Three Classical Settings:

1) Tilings by a set of tiles, in $\mathbb{E}^2$
1) Tilings by a set of tiles, in $\mathbb{R}^2$

In 1961, H. Wang noted connections between tiling problems and certain questions in formal logic. He gave an easy proof that the "Completion Problem" is undecidable, that is, that there is no algorithm to decide whether a given set of tiles can form a tiling of the plane containing a given configuration.
1) Tilings by a set of tiles, in $\mathbb{R}^2$

In 1961, H. Wang noted connections between tiling problems and certain questions in formal logic. He gave an easy proof that the "Completion Problem" is undecidable, that is, that there is no algorithm to decide whether a given set of tiles can form a tiling of the plane containing a given configuration.

He constructed, for any Turing machine, a set of tiles $T$ so that a certain "seed" configuration could be completed to a tiling if and only if the machine fails to halt. Since the Halting Problem is undecidable, so too is the Completion Problem.
1) Tilings by a set of tiles, in $\mathbb{R}^2$

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Note that the Domino Problem itself is decidable for the tiles Wang constructed.
In addition, Wang gave the algorithm we outlined above, for tilings by a set of tiles in the Euclidean plane, and incorrectly conjectured that the Domino Problem, in this setting, was decidable; it was difficult to imagine that an aperiodic set of tiles could exist.
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(Again, aperiodicity is an amazing property: the tiles do admit tilings, but somehow, just by fitting together locally, *force* some sort of disorder at all scales.)
In 1966, R. Berger published a proof that the Domino Problem is in fact undecidable, and gave the first aperiodic set of tiles. (In a moment we’ll discuss how.)
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Gurevich & Koryakov (1972) modified Berger’s construction to show the Period Problem, for planar sets of tiles, is undecidable as well.
One might make the case that the undecidability of these tiling problems, for planar sets of tiles, is the basis for a type of popular puzzle:
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Triazzles
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b-dazzles
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Scuzzles
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Scuzzles

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**Scuzzles**

These puzzles are “amusing” precisely because they have no logical structure—essentially every possibility must be examined.
Berger’s initial aperiodic set of tiles was notoriously complicated; subsequently, simpler aperiodic sets were found:

The Robinson tiles (1971) were the first reasonably small aperiodic set:
1) Every tile is either a □ or incident to □

2) Can't have: Only:

Hence:

3) So each □ is part of:

Hence, up to rotation, every tile is in or next to:
4) These 3x3 blocks act like large \( \square \)'s

& up to rotation, every tile is in or next to a 7x7 block:

& up to rotation, every tile is in or next to a 15x15 block, a 31x31 block, etc...

Consider a tiling by the Robinson tiles. Any translation has a finite magnitude and will translate some giant block onto itself. But this will not leave the tiling invariant. Hence every tiling by the Robinson tiles is non-periodic and the tiles themselves are aperiodic.
(incidentally, this hierarchical framework provides the key to Berger’s proof of the undecidability of the Domino Problem; in effect, he runs Wang’s implemented Turing machines on larger and larger domains in the hierarchy)
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The Penrose (-Ammann-Conway) tiles (1972-78) are the most famous example:
Berger’s initial aperiodic set of tiles was notoriously complicated; subsequently, simpler aperiodic sets were found:

Ammann gave several examples (ca. 1978), including:
Berger’s initial aperiodic set of tiles was notoriously complicated; subsequently, simpler aperiodic sets were found:

The trilobite and crab (GS, 1994):
Berger’s initial aperiodic set of tiles was notoriously complicated; subsequently, simpler aperiodic sets were found:

The Kari (Culik) (1995-6) tiles are the smallest known set of aperiodic Wang-tiles:
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The Penrose (-Socolar-GS) tiles (1994-96):
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What are the constraints on small aperiodic sets of tiles?
Berger’s initial aperiodic set of tiles was notoriously complicated; subsequently, simpler aperiodic sets were found:

What are the constraints on small aperiodic sets of tiles?  
Who knows?
For sets of tiles in the Euclidean plane, we have:

- Isohedral number is unbounded
- The Period Problem is undecidable (Berger 1966, others 1966–)
- Heesch number is unbounded
- The Domino Problem is undecidable (Berger 1966)

There is an aperiodic set of tiles (Berger 1966, others 1966–)
For sets of tiles in the Euclidean plane, we have:

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- Heesch number is unbounded
- The Period Problem is undecidable (Gurevich, Koryakov, 1972)
- The Domino Problem is undecidable (Berger 1966)
- There is an aperiodic set of tiles (Berger 1966 others 1966–)

A simple modification of Berger’s construction shows the Domino problem is undecidable for sets of tiles in $\mathbb{E}^n$ and $\mathbb{H}^{m \geq 3}$
2) Tilings by a single tile, in $\mathbb{E}^n$

In 1988, Schmitt gave an example of an aperiodic monotile in $\mathbb{E}^3$, later simplified by Conway and Danzer in 1993.

The SCD tile can only form sheets, that must be rotated as they stack.
2) Tilings by a single tile, in $\mathbb{E}^n$

In 1988, Schmitt gave an example of an aperiodic monotile in $\mathbb{E}^3$, later simplified by Conway and Danzer in 1993.

The SCD tile can only form sheets, that must be rotated as they stack.

The tile does not form a tiling with a compact fundamental domain. But it can form a tiling with an infinite cyclic action—that is, a period.
In the plane, it happens that if a set of tiles admits a tiling with a period, it admits a tiling with a finite fundamental domain.
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The SCD illustrates that this is not the case in other settings. In general, then, we might distinguish:
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A set of tiles is **weakly aperiodic** if it admits tilings, but no tiling with a compact fundamental domain.
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A set of tiles is **strongly aperiodic** if it admits tilings, but no tiling with an infinite cyclic action.
In the plane, it happens that if a set of tiles admits a tiling with a period, it admits a tiling with a finite fundamental domain.

The SCD illustrates that this is not the case in other settings. In general, then, we might distinguish:

A set of tiles is **weakly aperiodic** if it admits tilings, but no tiling with a compact fundamental domain.

A set of tiles is **strongly aperiodic** if it admits tilings, but no tiling with an infinite cyclic action.

It is open whether there is a strongly aperiodic monotile in $\mathbb{E}^3$. 
Very recently, Socolar showed that the isohedral number is unbounded for monotiles in $\mathbb{E}^3$:

These tiles can only form hexagonal parquets, as shown, which then stack to fill out space. The tile shown has isohedral number 5, which of course generalizes.
For monotiles in $\mathbb{E}^n$, we have:

- Isohedral number is unbounded (Socolar 2006)
- Heesch number is unbounded
- The Period Problem is undecidable
- The Domino Problem is undecidable
- There is a weakly aperiodic set of tiles (Schmitt 1988)
- There is a strongly aperiodic set of tiles
For **monotiles** in $\mathbb{E}^n$, we have:

- There is a weakly aperiodic set of tiles (Schmitt 1988)
- There is a strongly aperiodic set of tiles
- The Domino Problem is undecidable (Schmitt 1988)
- The Period Problem is undecidable
- Isohedral number is unbounded (Socolar 2006)
- Heesch number is unbounded

Both the SCD and Socolar’s constructions are quite elegant — perhaps there are very simple examples that settle these remaining questions?
3) Sets of tiles in $\mathbb{H}^2$
In the hyperbolic plane, it is not difficult to find weakly aperiodic tiles:
3) Sets of tiles in $\mathbb{H}^2$
In the hyperbolic plane, it is not difficult to find weakly aperiodic tiles: For example,

(GS 2001) *Almost every triangle that does admit a tiling of $\mathbb{H}^2$ is weakly aperiodic!*
Almost every triangle that does admit a tiling of $\mathbb{H}^2$ is weakly aperiodic!
However, weakly aperiodic sets of tiles seem too simple. In (2005) the first strongly aperiodic set of tiles was found in the hyperbolic plane (GS), modifying Kari’s construction in $\mathbb{E}^2$. 

![Diagram of aperiodic set of tiles]
However, weakly aperiodic sets of tiles seem too simple. In (2005) the first strongly aperiodic set of tiles was found in the hyperbolic plane (GS), modifying Kari’s construction in $\mathbb{H}^2$.

These tiles admit only tilings with no symmetry whatsoever!
Thus, for **sets of tiles in** $\mathbb{H}^2$, **we would have**:

- There is a weakly aperiodic set of tiles (folk <1977)
- There is a strongly aperiodic set of tiles (GS 2005)
- The Domino Problem is undecidable
- The Period Problem is undecidable
- Isohedral number is unbounded
- Heesch number is unbounded
However just now, at this conference, we have two announced proofs, by Kari and by Margenstern, that the Domino Problem is undecidable in $\mathbb{H}^2$!

- There is a weakly aperiodic set of tiles (folk <1977)
- There is a strongly aperiodic set of tiles (GS 2005)

- The Period Problem is undecidable
- Isohedral number is unbounded
- Heesch number is unbounded

(Kari 2007, Margenstern 2007)
You may keep the toys...

Other goodies available up front