Puzzles

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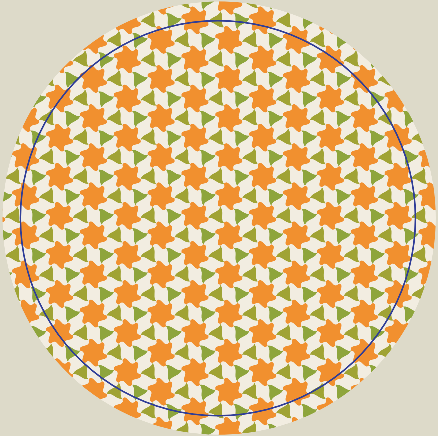
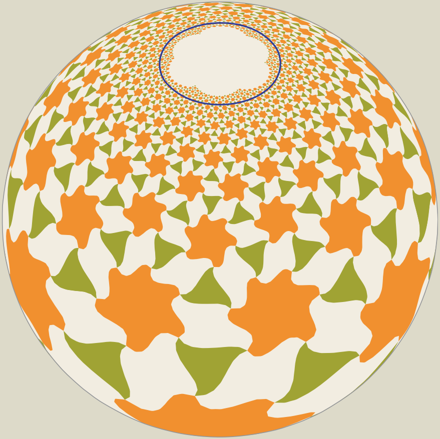
Davidson, NC
March 2007

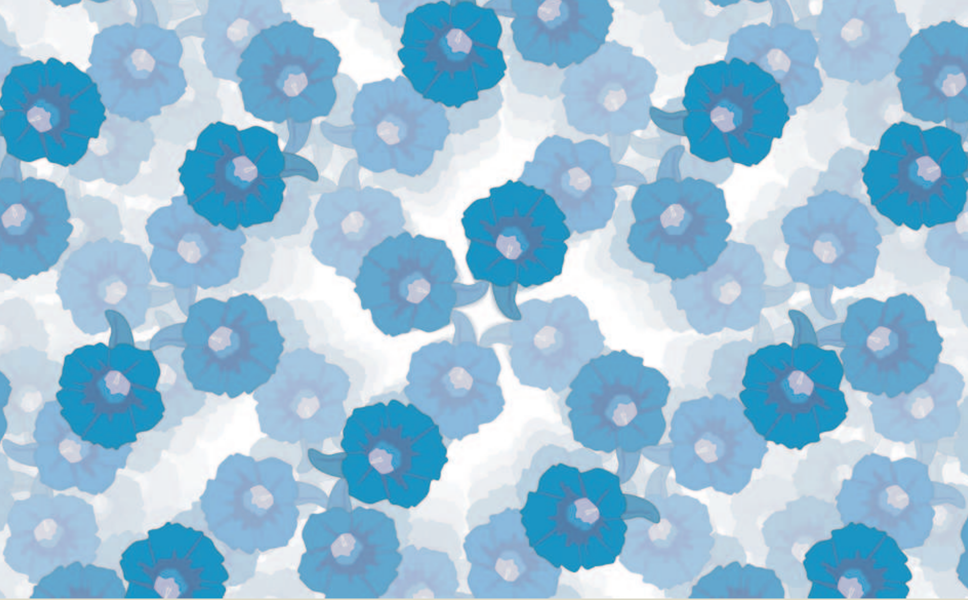
A quick advertisement for the forthcoming compendium





The Symmetry of Things

with John H. Conway, Heidi Burgiel (2007)



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Math Factor Podcast

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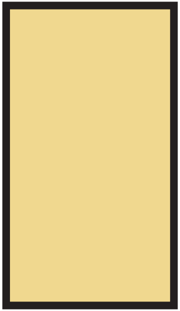
Can this tile be used to form a tiling of the entire plane?

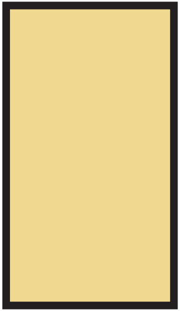


trivial example

We begin with a question:

Can this tile be used to form a tiling of the entire plane?

How can you tell?



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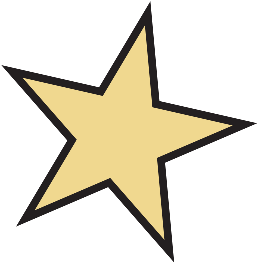

certainly not

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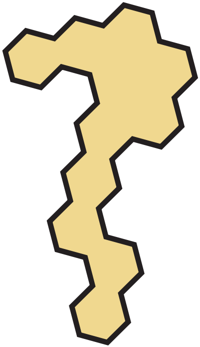

Myers (2003)

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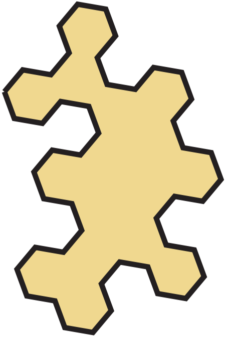

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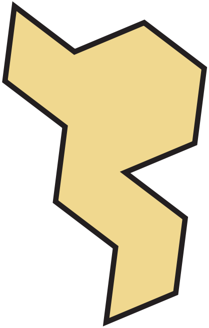

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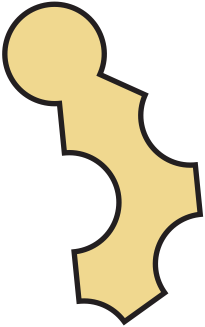

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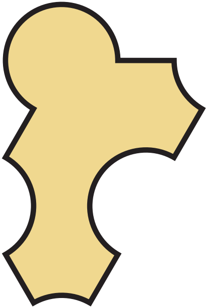

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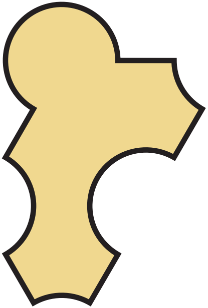
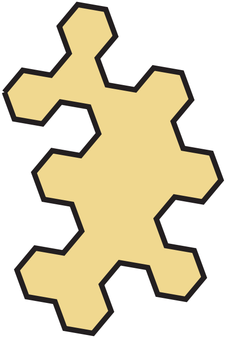


Mann (2007)

Is there a general method to tell whether or not a given tile admits a tiling?

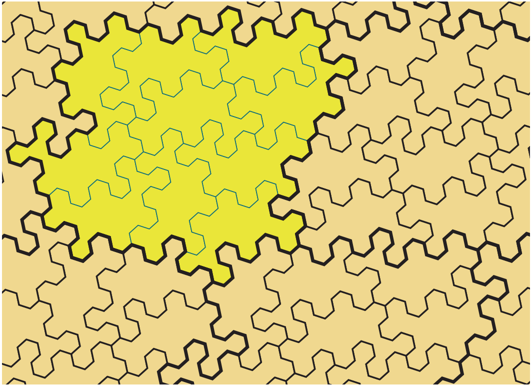
Many of these examples are very badly behaved.

This example (Myers 2003) has *isohedral number* 10, the current world record



Many of these examples are very badly behaved.

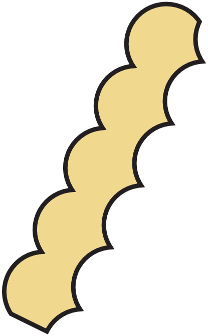
That is, the tile can form periodic tilings, but the tiles fall into at least ten orbits; equivalently, the smallest possible fundamental domain has ten tiles!



An isohedral number 10 example

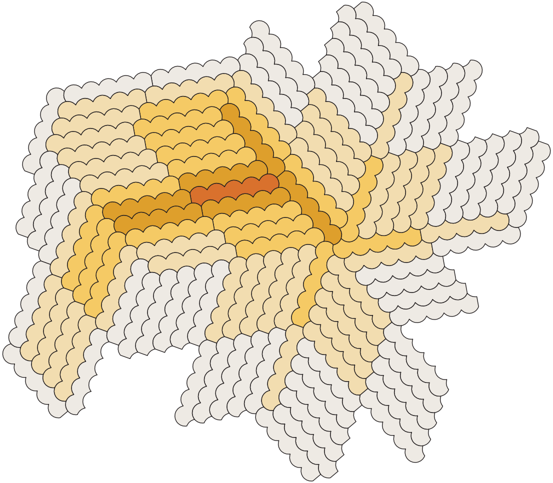
Many of these examples are very badly behaved.

This example (Mann 1999) doesn't admit a tiling, but you can form quite large patches before things fall apart.



Many of these examples are very badly behaved.

It has *Heesch number* 5, the current world record– it can form a patch with five “coronas" and no more.



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Again, *Is there a way to tell whether a given tile admits a tiling of the entire plane?*

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See how far you can get!

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• If the tile does *not* admit a tiling, then there will be some largest region that we cannot cover.

Thm: If we can cover arbitrarily large regions, we can cover the entire plane.

The obvious algorithm to try is
See how far you can get!

More precisely, we might enumerate all possible configurations by the tile, trying to cover larger and larger regions.

• If the tile does *not* admit a tiling, then there will be some largest region that we cannot cover.

But this algorithm will never halt if the tile *does* admit a tiling.

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• If the tile does *not* admit a tiling, then there will be some largest region that we cannot cover.

But this algorithm will never halt if the tile *does* admit a tiling.
So we make a modification.

The obvious algorithm to try is
See how far you can get!

More precisely, we might enumerate all possible configurations by the tile, trying to cover larger and larger regions.

• If the tile does *not* admit a tiling, then there will be some largest region that we cannot cover.

• As we proceed, we check to see if any of our configurations could be a fundamental domain. If the tile admits a *periodic* tiling, we can discover this too.

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See how far you can get!

More precisely, we might enumerate all possible configurations by the tile, trying to cover larger and larger regions.

• If the tile does *not* admit a tiling, then there will be some largest region that we cannot cover.

• As we proceed, we check to see if any of our configurations could be a fundamental domain. If the tile admits a *periodic* tiling, we can discover this too.

This works fine— the algorithm will halt with a yes or no answer, so long as nothing falls through the gaps— so long as every tile either doesn't admit a tiling or admits a periodic tiling.

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How hard can this be?

The examples we see here should make us cautious.

The algorithm we outlined (enumerate all configurations, covering larger and larger disks, until we either crash, or find a fundamental domain) succeeds if there is no aperiodic tile.

If there is no aperiodic tile, then this algorithm solves both the Domino Problem and the Period Problem for any given monotile:

The Domino Problem for Monotiles:

Given a tile, does it fail to admit a tiling?

The Period Problem for Monotiles:

Given a tile, does it admit a periodic tiling?

That is:

The Period Problem is decidable for monotiles $\Leftarrow$ There is no aperiodic monotile

There is no aperiodic monotile $\Rightarrow$ The Domino Problem is decidable for monotiles

That is:

The Period Problem is undecidable for monotiles ⇘

There is an aperiodic monotile

The Domino Problem is undecidable for monotiles ⇗

Moreover, suppose there is a bound H on Heesch number; i.e., suppose every tile that does not admit a tiling has Heesch number less than H.

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Then we would have an algorithm for checking whether a given tile does not admit a tiling:

Simply enumerate larger and larger configurations, increasing the number of coronas (shells). If we find a configuration with more than H coronas, we know the tile admits a tiling.

Similarly, suppose there is a bound I on isohedral number; i.e. suppose that every tile that admits a periodic tiling has isohedral number less than I.

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Then we would have an algorithm for checking whether a given tile admits a periodic tiling:

Simply enumerate all configurations with up to I tiles; if we fail to find a fundamental domain, then the tile does not admit a periodic tiling.

And so we have

$$
\begin{tikzcd}
\text{Isohedral number is unbounded for monotiles} & \text{The Period Problem is undecidable for monotiles} \arrow[l, Rightarrow] \arrow[dr, Rightarrow] & \\
& & \text{There is an aperiodic monotile} \\
\text{Heesch number is unbounded for monotiles} & \text{The Domino Problem is undecidable for monotiles} \arrow[l, Rightarrow] \arrow[ur, Rightarrow] &
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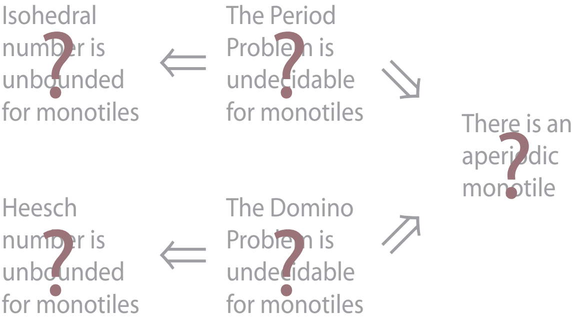
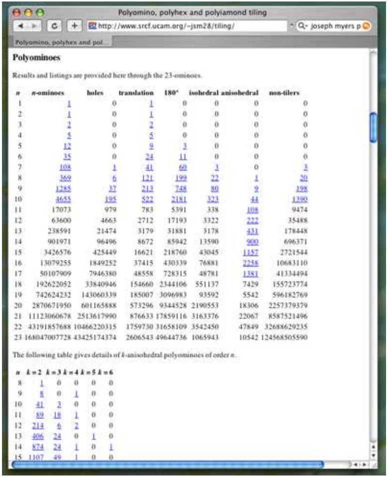


These are all open questions.

Myers, Mann and others are generating increasingly complex examples, via giant computer searches, that suggest that the Domino problem may well be undecidable for monotiles, and that there does exist an aperiodic monotile.



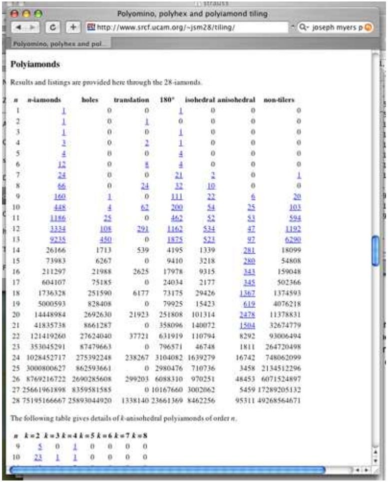
Polyominoes

Results and listings are provided here through the 23-ominoes.

n	*n*-ominoes	holes	translation	180°	isohedral	anisohedral	non-tilers
1	1	0	1	0	0	0	0
2	1	0	1	0	0	0	0
3	2	0	2	0	0	0	0
4	5	0	5	0	0	0	0
5	12	0	9	3	0	0	0
6	35	0	24	11	0	0	0
7	108	1	41	60	3	0	3
8	369	6	121	199	22	1	20
9	1285	37	213	748	80	9	198
10	4655	195	522	2181	323	44	1390
11	17073	979	783	5391	338	108	9474
12	63600	4663	2712	17193	3322	222	35488
13	238591	21474	3179	31881	3178	431	178448
14	901971	96496	8672	85942	13590	900	696371
15	3426576	425449	16621	218760	43045	1157	2721544
16	13079255	1849252	37415	430339	76881	2258	10683110
17	50107909	7946380	48558	728315	48781	1381	41334494
18	192622052	33840946	154660	2344106	551137	7429	155723774
19	742624232	143060339	185007	3096983	93592	5542	596182769
20	2870671950	601165888	573296	9344528	2190553	18306	2257379379
21	11123060678	2513617990	876633	17859116	3163376	22067	8587521496
22	43191857688	10466220315	1759730	31658109	3542450	47849	32688629235
23	168047007728	43425174374	2606543	49644736	1065943	10542	124568505590

The following table gives details of *k*-anisohedral polyominoes of order *n*.

n	*k* = 2	*k* = 3	*k* = 4	*k* = 5	*k* = 6
8	1	0	0	0	0
9	8	0	1	0	0
10	41	3	0	0	0
11	89	18	1	0	0
12	214	6	2	0	0
13	406	24	0	1	0
14	874	24	1	0	1
15	1107	49	1	0	0



Polyiamonds

Results and listings are provided here through the 28-iamonds.

n	*n*-iamonds	holes	translation	180°	isohedral	anisohedral	non-tilers
1	1	0	0	1	0	0	0
2	1	0	1	0	0	0	0
3	1	0	0	1	0	0	0
4	3	0	2	1	0	0	0
5	4	0	0	4	0	0	0
6	12	0	8	4	0	0	0
7	24	0	0	21	2	0	1
8	66	0	24	32	10	0	0
9	160	1	0	111	22	6	20
10	448	4	62	200	54	25	103
11	1186	25	0	462	52	53	594
12	3334	108	291	1162	534	47	1192
13	9235	450	0	1875	523	97	6290
14	26166	1713	539	4195	1339	281	18099
15	73983	6267	0	9410	3218	280	54808
16	211297	21988	2625	17978	9315	343	159048
17	604107	75185	0	24034	2177	345	502366
18	1736328	251590	6177	73175	29426	1367	1374593
19	5000593	828408	0	79925	15423	619	4076218
20	14448984	2692630	21923	251808	101314	2478	11378831
21	41835738	8661287	0	358096	140072	1504	32674779
22	121419260	27624040	37721	631919	110794	8292	93006494
23	353045291	87479663	0	796571	46748	1811	264720498
24	1028452717	275392248	238267	3104082	1639279	16742	748062099
25	3000800627	862593661	0	2980476	710736	3458	2134512296
26	8769216722	2690285608	299203	6088310	970251	48453	6071524897
27	25661961898	8359581585	0	10167660	3002062	5459	17289205132
28	75195166667	25893044920	1338140	23661369	8462256	95311	49268564671

The following table gives details of *k*-anisohedral polyiamonds of order *n*.

n	*k* = 2	*k* = 3	*k* = 4	*k* = 5	*k* = 6	*k* = 7	*k* = 8
9	5	0	1	0	0	0	0
10	23	1	1	0	0	0	0

Myers, Mann and others are generating increasingly complex examples, via giant computer searches, that suggest that the Domino problem may well be undecidable for monotiles, and that there does exist an aperiodic monotile.

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But as with all such problems

(Is Question A decidable for combinatorial objects in X)

the searches are increasingly intractable, and the generic example does what it does for no particular reason.

(A “reason" is essentially an algorithm for deciding something!)

These tiles can be marketed as children’s toys, yet are an example of Undecidability in action!

These same questions can be asked in a range of settings:

Generally speaking, we specify a geometric space, and combinatorial restrictions on what we mean by “tile", “matching rule", etc.

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For example we might consider:

Tilings of $\mathbb{E}^2$, by a single tile (as we’ve just considered)

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Generally speaking, we specify a geometric space, and combinatorial restrictions on what we mean by “tile", “matching rule", etc.

For example we might consider:

(more specifically: Tilings of $\mathbb{E}^2$, by a single polygonal disk)

These same questions can be asked in a range of settings:

Generally speaking, we specify a geometric space, and combinatorial restrictions on what we mean by “tile", “matching rule", etc.

For example we might consider:

Tilings of $\mathbb{E}^2$, by a set of tiles

These same questions can be asked in a range of settings:

Generally speaking, we specify a geometric space, and combinatorial restrictions on what we mean by “tile", “matching rule", etc.

For example we might consider:

Tilings of $\mathbb{H}^2$ by . . .

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Generally speaking, we specify a geometric space, and combinatorial restrictions on what we mean by “tile", “matching rule", etc.

For example we might consider:

Tilings of $\mathbb{E}^2$ by a shingle

These same questions can be asked in a range of settings:

Generally speaking, we specify a geometric space, and combinatorial restrictions on what we mean by “tile'', “matching rule'', etc.

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Tilings of $\mathbb{E}^2$ by a disconnected tile

Each of these is strikingly different, and we must be explicit about which setting we are considering!

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Generally speaking, we specify a geometric space, and combinatorial restrictions on what we mean by “tile", “matching rule", etc.

For example we might consider:

Tilings of $\mathbb{E}^2$, by tiles that have colored edges (colors must match)

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These same questions can be asked in a range of settings:

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For example we might consider:

Tilings of $\mathbb{E}^2$, by tiles that have colored edges, but satisfy an arbitrary relation (for example, red may meet with green or blue, but green may not meet blue)

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Tilings of $\mathbb{E}^3$, by a single tile

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Etc.

Each of these is strikingly different, and we must be explicit about which setting we are considering!

In any given specified setting, in which we can enumerate configurations we have:

Isohedral number is unbounded	$\Leftarrow$	The Period Problem is undecidable	$\Searrow$	
				There is an aperiodic set of tiles
Heesch number is unbounded	$\Leftarrow$	The Domino Problem is undecidable	$\Nearrow$	

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\text{Heesch number is unbounded} & \text{The Domino Problem is undecidable} \arrow[l, Rightarrow] \arrow[ur, Rightarrow] &
\end{tikzcd}$$

(And there are more, similar kinds of implications)

Three Classical Settings:

1) Tilings by a set of tiles, in $\mathbb{E}^2$

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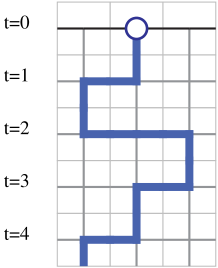
He constructed, for any Turing machine, a set of tiles T so that a certain "seed" configuration could be completed to a tiling if and only if the machine fails to halt. Since the Halting Problem is undecidable, so too is the Completion Problem.



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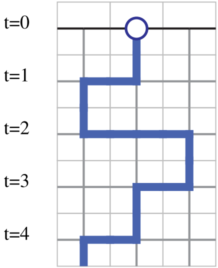

Note that the Domino Problem itself *is* decidable for the tiles Wang constructed.

In addition, Wang gave the algorithm we outlined above, for tilings by a set of tiles in the Euclidean plane, and incorrectly conjectured that the Domino Problem, in this setting, was decidable; it was difficult to imagine that an aperiodic set of tiles could exist.

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(Again, aperiodicity is an amazing property: the tiles do admit tilings, but somehow, just by fitting together locally, *force* some sort of disorder at all scales.)

In 1966, R. Berger published a proof that the Domino Problem is in fact undecidable, and gave the first aperiodic set of tiles. (In a moment we’ll discuss how.)

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Gurevich & Koryakov (1972) modified Berger’s construction to show the Period Problem, for planar sets of tiles, is undecidable as well.

One might make the case that the undecidability of these tiling problems, for planar sets of tiles, is the basis for a type of popular puzzle:

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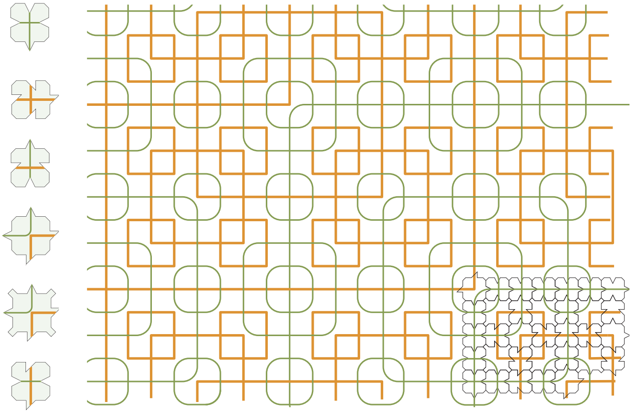
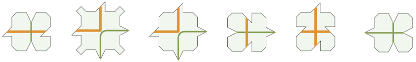
Berger's initial aperiodic set of tiles was notoriously complicated; subsequently, simpler aperiodic sets were found:

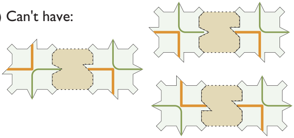
The Robinson tiles (1971) were the first reasonably small aperiodic set:



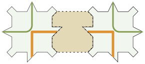


1) Every tile is either a  or incident to 

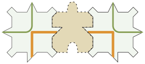
2) Can't have:



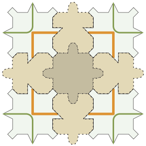
Only:



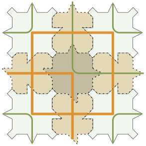
Hence:



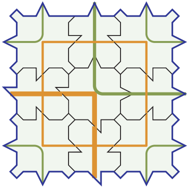
3) So each  is part of:



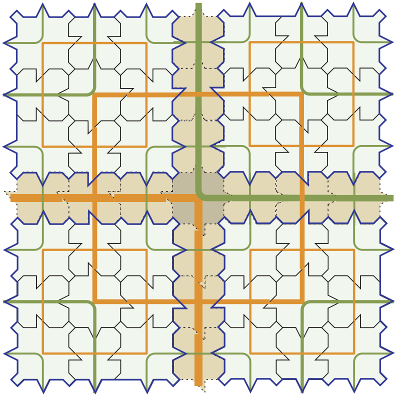
Hence, up to rotation,
every tile is in or next to:



4) These 3x3 blocks act like large  's



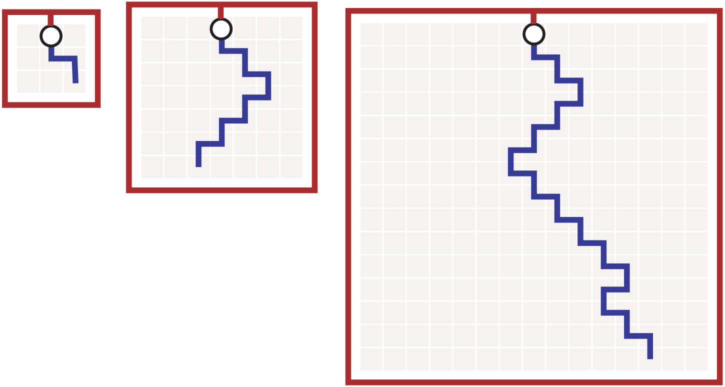
& up to rotation,
every tile is in or next to
a 7x7 block:



& up to rotation, every tile is in or next to a 15x15 block, a 31x31 block, etc...

Consider a tiling by the Robinson tiles. Any translation has a finite magnitude and will translate some giant block onto itself. But this will not leave the tiling invariant. Hence every tiling by the Robinson tiles is non-periodic and the tiles themselves are aperiodic.

(incidentally, this hierarchical framework provides the key to Berger's proof of the undecidability of the Domino Problem; in effect, he runs Wang's implemented Turing machines on larger and larger domains in the hierarchy)



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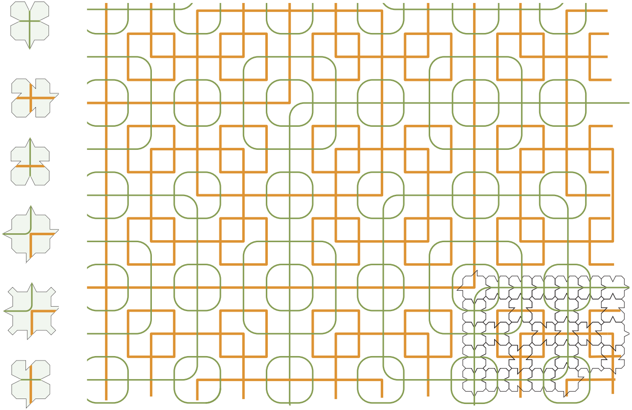
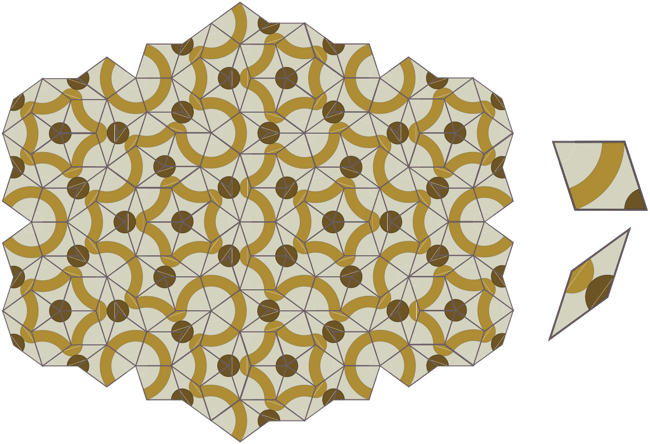


Berger’s initial aperiodic set of tiles was notoriously complicated; subsequently, simpler aperiodic sets were found:

The Penrose (-Ammann-Conway) tiles (1972-78) are the most famous example:



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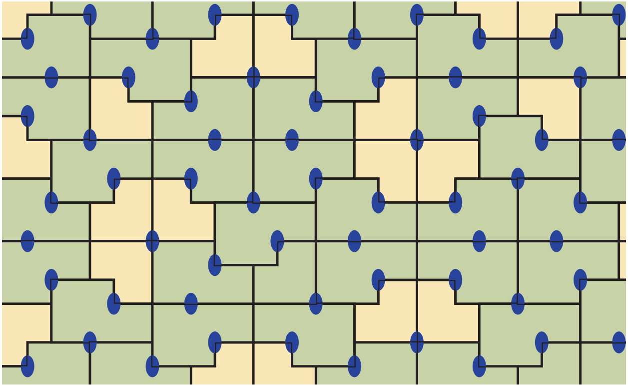
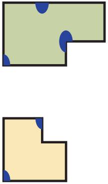
Ammann gave several examples (ca. 1978), including:





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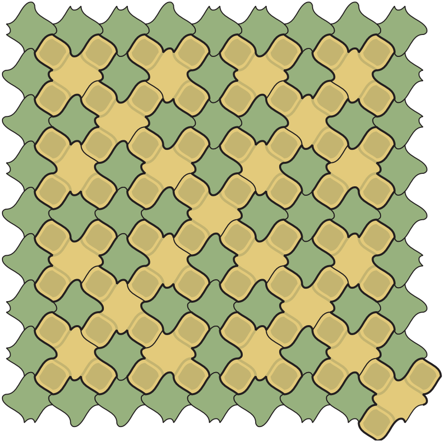
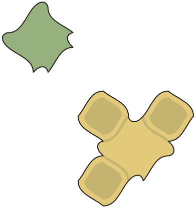
The trilobite and crab (GS, 1994):





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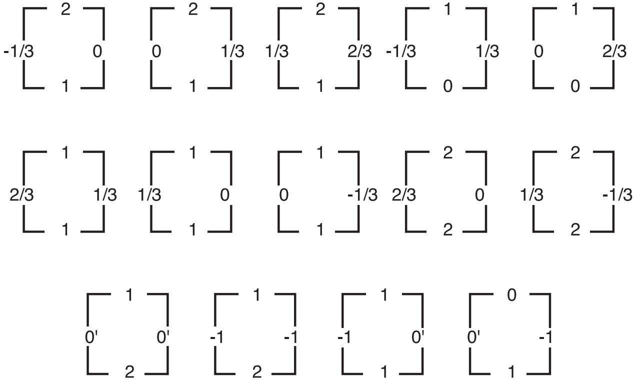

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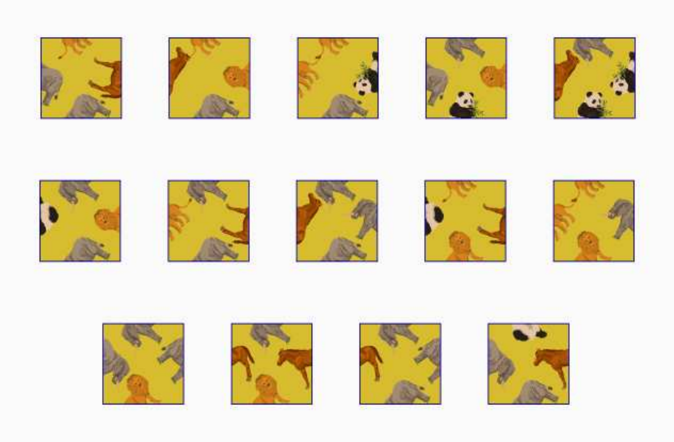

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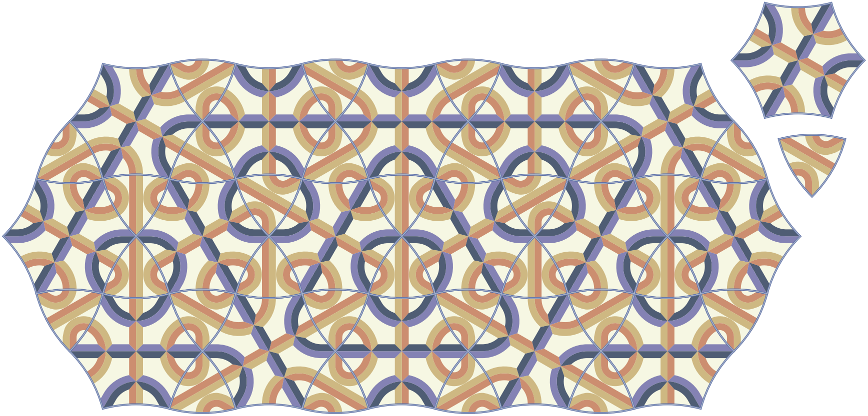
The Penrose (-Socolar-GS) tiles (1994-96):



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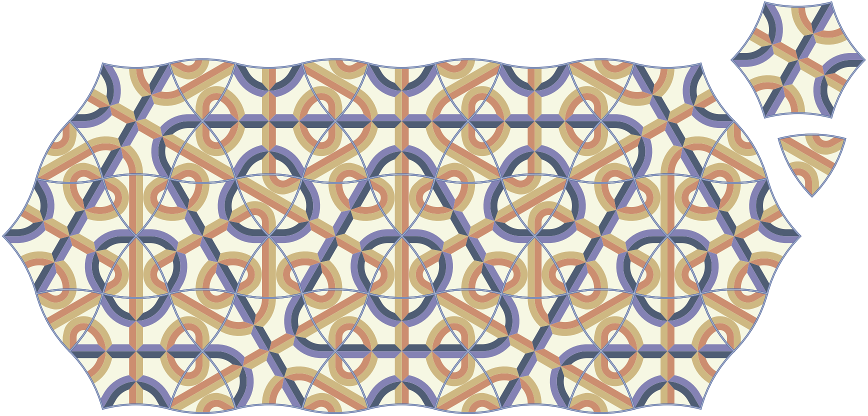

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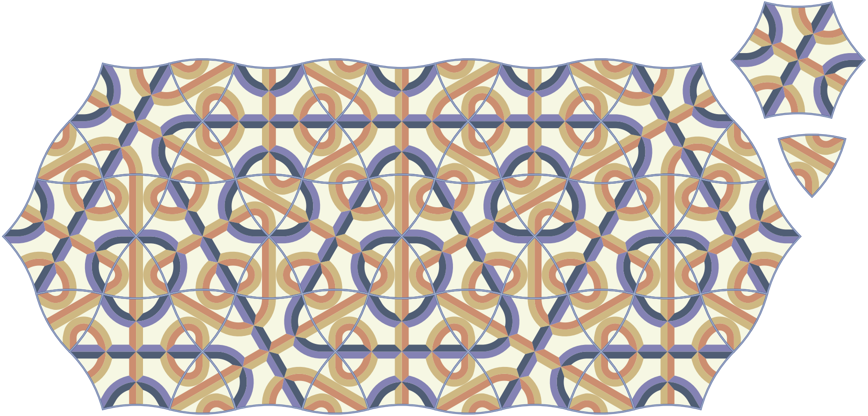

What are the constraints on small aperiodic sets of tiles?

Who knows?

For **sets of tiles in the Euclidean plane**, we have:

Isohedral number is unbounded $\Longleftarrow$ The Period Problem is undecidable (Gurevich, Koryakov, 1972) $\Searrow$

There is an aperiodic set of tiles (Berger 1966 others 1966–

Heesch number is unbounded $\Longleftarrow$ The Domino Problem is undecidable (Berger 1966) $\Nearrow$

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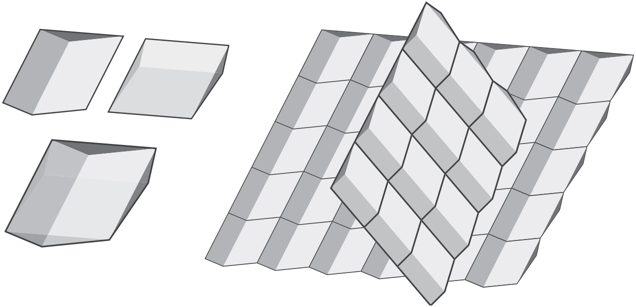
A simple modification of Berger's construction shows the Domino problem is undecidable for sets of tiles in $\mathbb{E}^n$ and $\mathbb{H}^{m\geq 3}$

2) Tilings by a single tile, in $\mathbb{E}^n$

In 1988, Schmitt gave an example of an aperiodic monotile in $\mathbb{E}^3$, later simplified by Conway and Danzer in 1993.

The SCD tile can only form sheets, that must be rotated as they stack.



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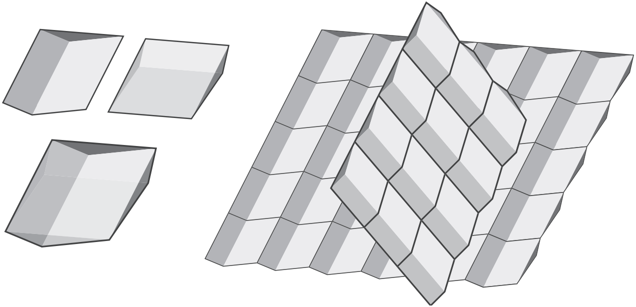

The tile does *not* form a tiling with a compact fundamental domain. But it *can* form a tiling with an infinite cyclic action—that is, a period.

In the plane, it happens that if a set of tiles admits a tiling with a period, it admits a tiling with a finite fundamental domain.

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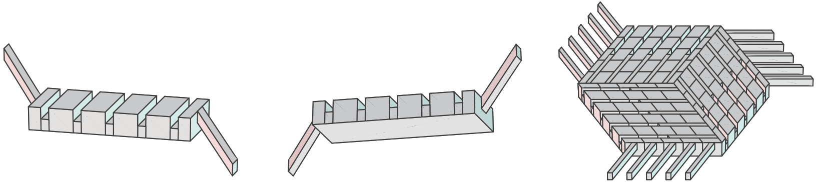
It is open whether there is a strongly aperiodic monotile in $\mathbb{E}^3$.

Very recently, Socolar showed that the isohedral number is unbounded for monotiles in $\mathbb{E}^3$:



These tiles can only form hexagonal parquets, as shown, which then stack to fill out space. The tile shown has isohedral number 5, which of course generalizes.

For **monotiles in $\mathbb{E}^n$**, we have:

Isohedral number is unbounded (Socolar 2006) $\Leftarrow$? The Period Problem is undecidable $\Searrow$

There is a weakly aperiodic set of tiles (Schmitt 1988) $\Leftarrow$? There is a strongly aperiodic set of tiles

? Heesch number is unbounded $\Leftarrow$? The Domino Problem is undecidable $\Nearrow$

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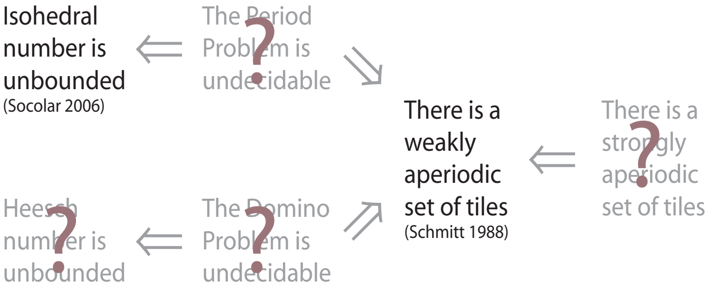
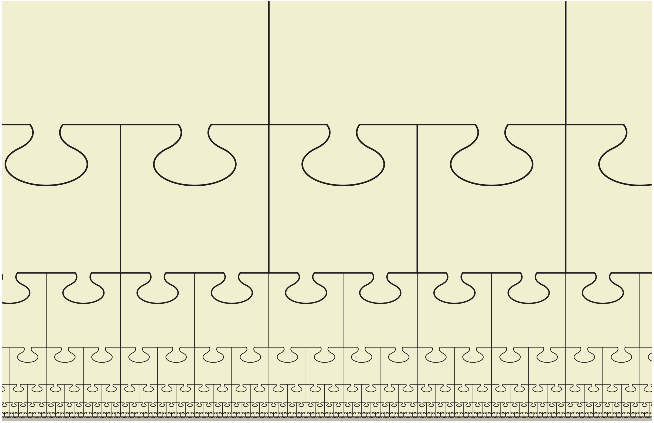


Both the SCD and Socolar’s constructions are quite elegant — perhaps there are very simple examples that settle these remaining questions?

3) Sets of tiles in $\mathbb{H}^2$

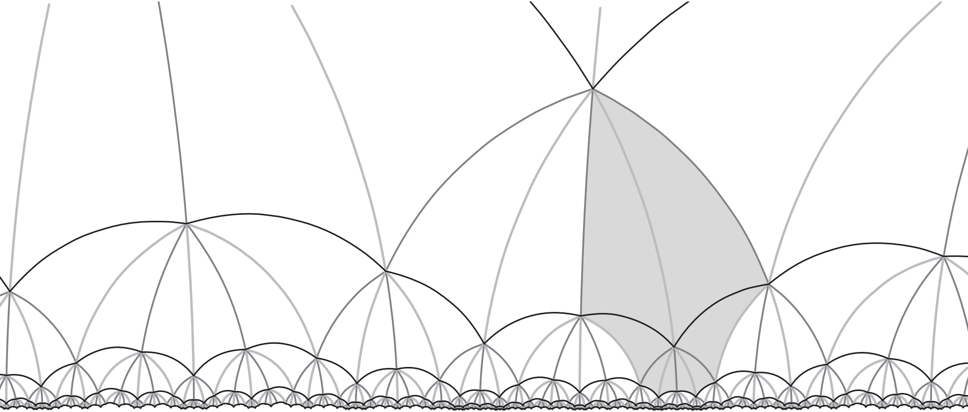
In the hyperbolic plane, it is not difficult to find weakly aperiodic tiles:



3) Sets of tiles in $\mathbb{H}^2$

In the hyperbolic plane, it is not difficult to find weakly aperiodic tiles: For example,

(GS 2001)*Almost every triangle that does admit a tiling of* $\mathbb{H}^2$ *is weakly aperiodic!*



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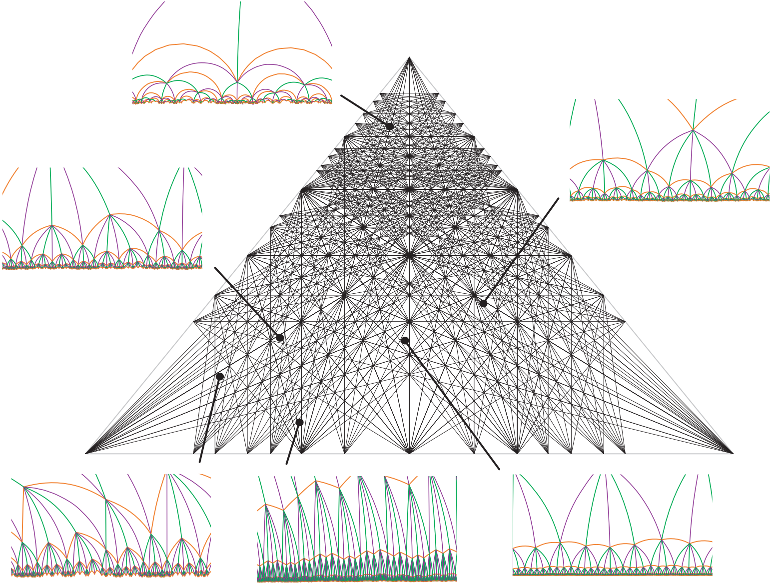
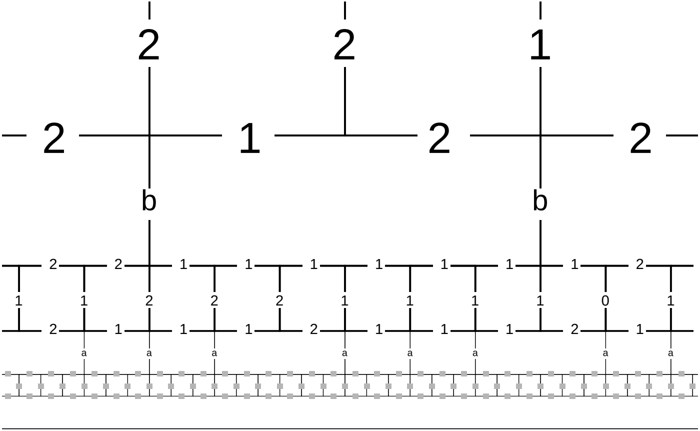


However, weakly aperiodic sets of tiles seem too simple. In (2005) the first strongly aperiodic set of tiles was found in the hyperbolic plane (GS), modifying Kari's construction in $\mathbb{E}^2$.



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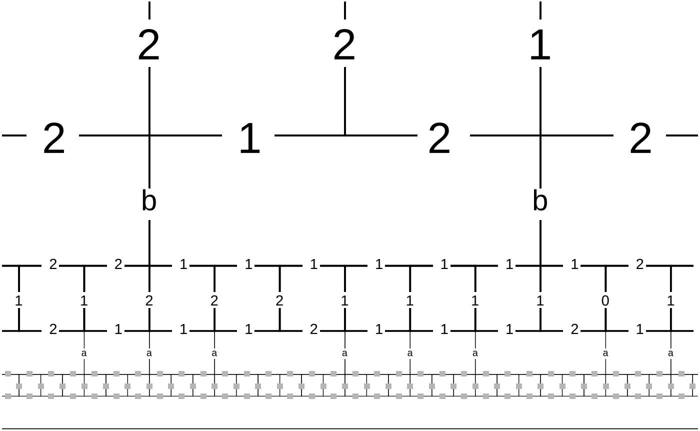

These tiles admit only tilings with no symmetry whatsoever!

Thus, for **sets of tiles in $\mathbb{H}^2$**, we would have:

$$
\begin{tikzcd}
\text{Isohedral number is unbounded ?} & \text{The Period Problem is undecidable ?} \arrow[l, Rightarrow] \arrow[dr, Rightarrow] & & \\
& & \text{There is a weakly aperiodic set of tiles (folk <1977)} & \text{There is a strongly aperiodic set of tiles (GS 2005)} \arrow[l, Rightarrow] \\
\text{Heesch number is unbounded ?} & \text{The Domino Problem is undecidable ?} \arrow[l, Rightarrow] \arrow[ur, Rightarrow] & &
\end{tikzcd}
$$

However just now, at this conference, we have two announced proofs, by Kari and by Margenstern, that the Domino Problem is undecidable in $\mathbb{H}^2$!

?Isohedral number is unbounded $\Leftarrow$?The Period Problem is undecidable $\Searrow$

There is a weakly aperiodic set of tiles (folk <1977) $\Leftarrow$ There is a strongly aperiodic set of tiles (GS 2005)

Heesch number is unbounded $\Leftarrow$ The Domino Problem is undecidable (Kari 2007 Margenstern 2007) $\Nearrow$

You may keep the toys...

Other goodies available up front