The Berger-Robinson-Margenstern Proof

London, August 1, 2008

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A hierarchical aperiodic set in H^2

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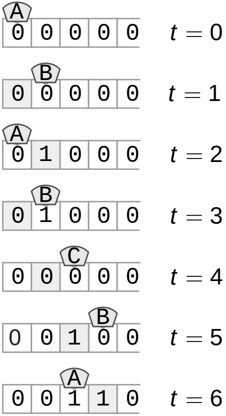
In 1961, H. Wang noted connections between tiling problems and certain questions in formal logic. He gave an easy proof that the "Completion Problem" is undecidable, that is, that there is no algorithm to decide whether a given set of tiles can form a tiling of the plane containing a given configuration.

Wang constructed, for any Turing machine, a set of tiles T so that a certain "seed" configuration could be completed to a tiling if and only if the machine fails to halt. Since the Halting Problem is undecidable, so too is the Completion Problem.

ϕ	A	B	C
0	0RB	1LA	1RB
1	1RB	0RC	0LH



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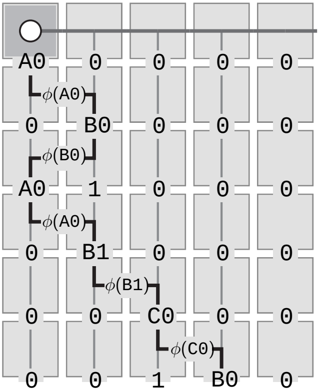

As an aside Wang remarks:

What appears to be a reasonable conjecture, which has resisted proof or disproof so far, is:

4.1.2 *The fundamental conjecture*: A finite set of plates is solvable (has at least one solution) if and only if there exists a cyclic rectangle of the plates; or, in other words, a finite set of plates is solvable if and only if it has at least one periodic solution.

It is easy to prove the following:

4.1.3 If 4.1.2 is true, we can decide effectively whether any given finite set of plates is solvable.

In other words (up to some minor conventions) Wang conjectured that if a set of tiles admits a tiling at all, then it admits a periodic tiling, and points out that if this is so, then the “Domino Problem" is decidable, that there is a procedure to decide whether a given set of tiles admits a tiling.

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Conversely, then, if there is no such procedure, there must exist *aperiodic* sets of tiles: tiles that admit tilings, but only non-periodic ones.

In 1964-66, Robert Berger in fact proved that the Domino Problem is undecidable. That is, there is no general procedure that can answer, for any given set of tiles, in finite time Yes or No:

Does this set of tiles admit a tiling of the plane?

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This style of proof is the subject of my talk today.

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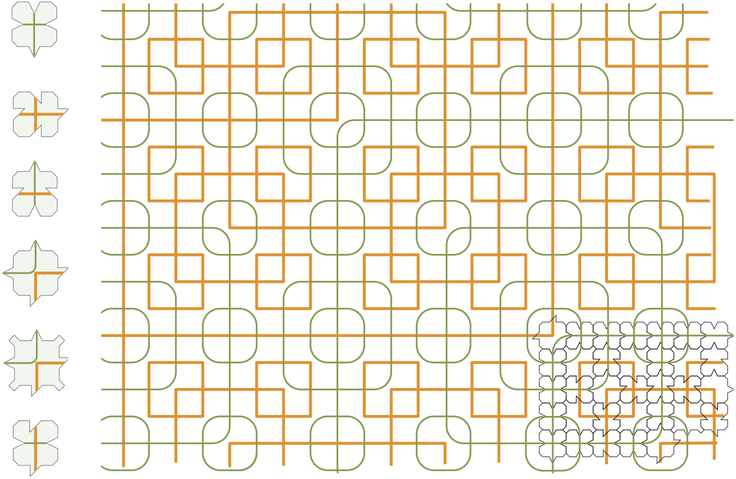
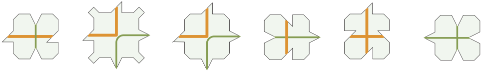
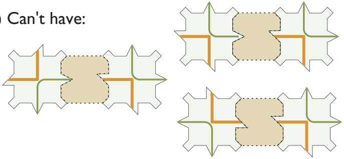
Consider the following set of six tiles: Not only does the set admit non-periodic tilings, it *only* does so and is thus *aperiodic*.



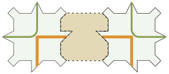


1) Every tile is either a  or incident to 

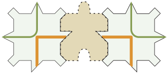
2) Can't have:



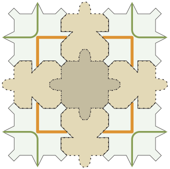
Only:



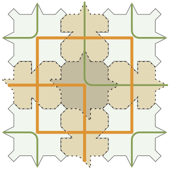
Hence:



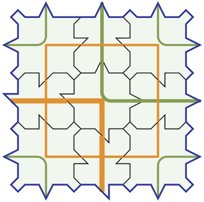
3) So each  is part of:



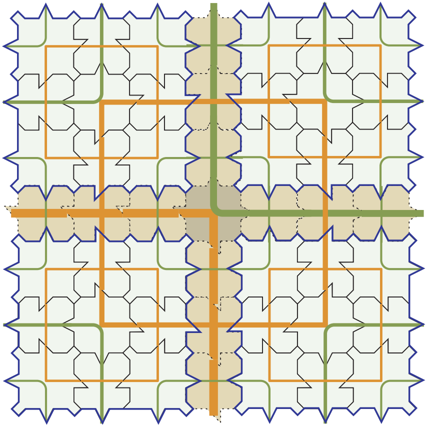
Hence, up to rotation,
every tile is in or next to:



4) These 3x3 blocks act like large  's



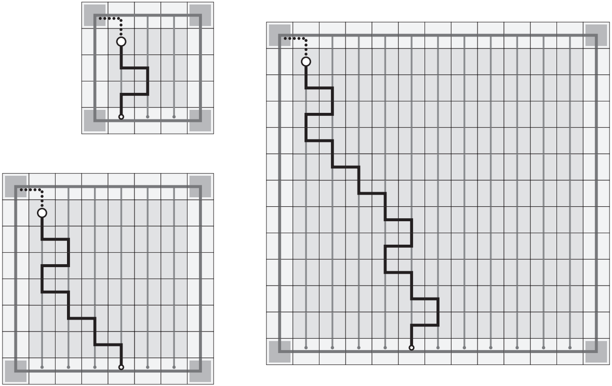
& up to rotation,
every tile is in or next to
a 7x7 block:



& up to rotation, every tile is in or next to a 15x15 block, a 31x31 block, etc...

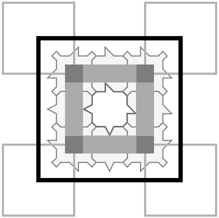
Consider a tiling by the Robinson tiles. Any translation has a finite magnitude and will translate some giant block onto itself. But this will not leave the tiling invariant. Hence every tiling by the Robinson tiles is non-periodic and the tiles themselves are aperiodic.

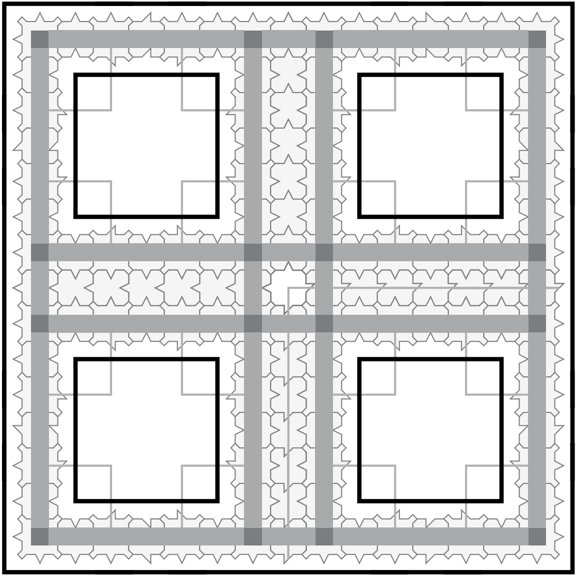
Berger and Robinson make very strong use of the hierarchical nature of an underlying aperiodic set of tiles; the basic idea is to run Wang's implemented Turing machines on larger and larger domains in the hierarchy.

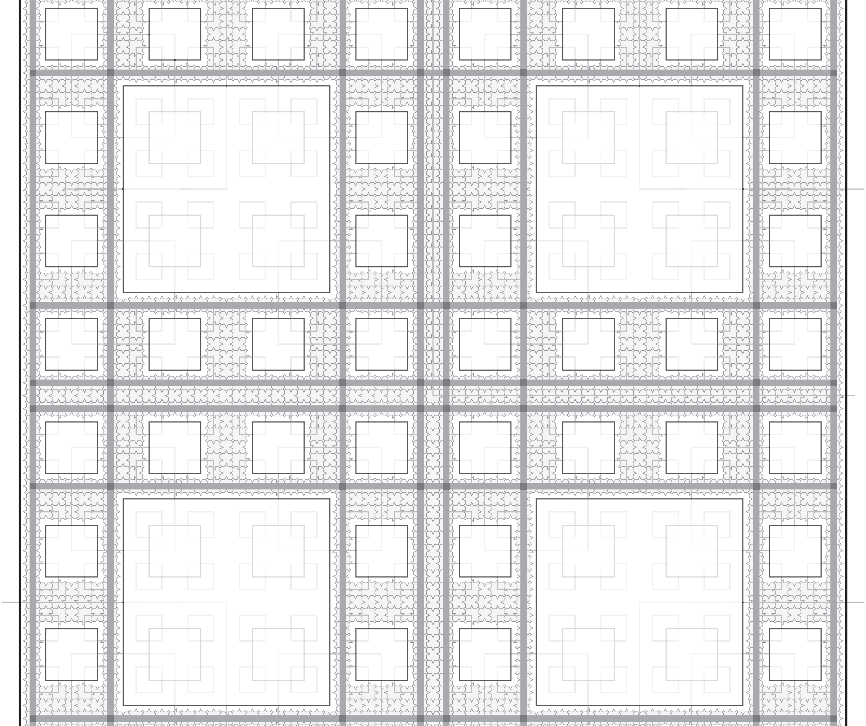


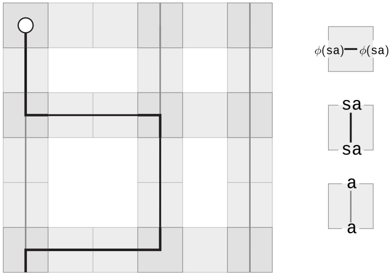
The tiles Berger and Robinson design are to *force* the appearance of arbitrarily large domains in any tiling they admit (and in each domain, force the “seed" tile that begins the emulation).

Their aperiodic sets of tiles *force* a hierarchical structure which can be exploited to this end:





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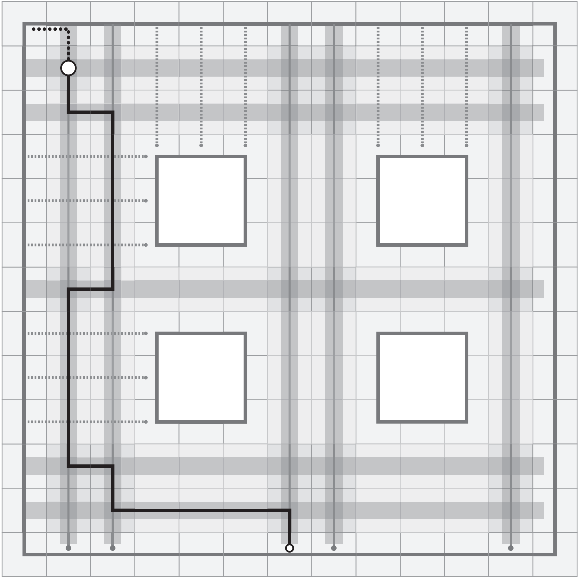
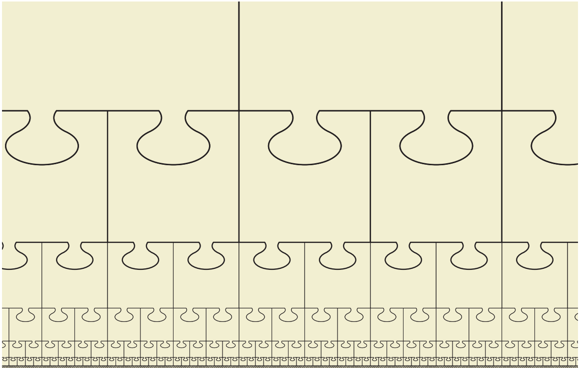


In other settings

Implicitly we have been limiting the context of our discussion to . . . , . . . tiles in the Euclidean plane. If we are careful to specify the setting we can consider these questions in other contexts.

For example, ca. 1977, Penrose noted there are aperiodic sets of tiles in the hyperbolic plane.



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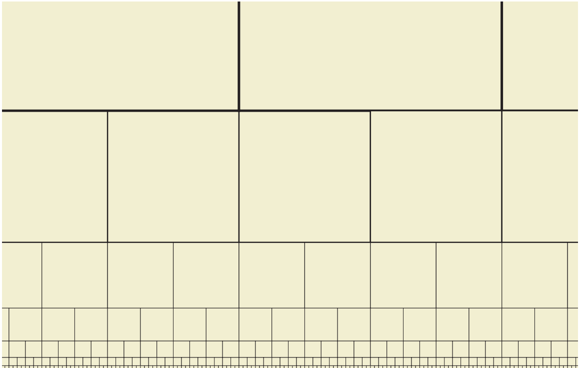

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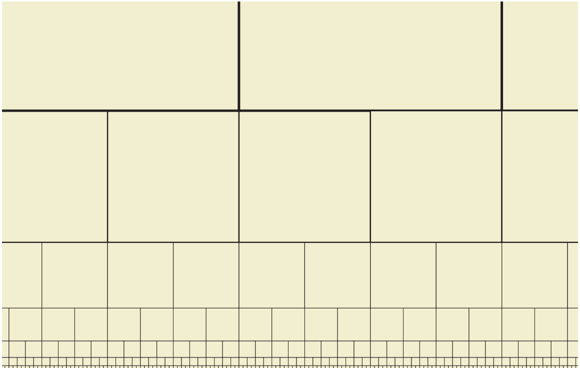

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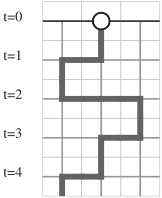

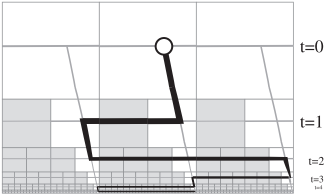

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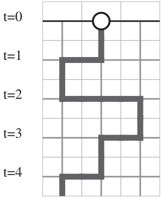

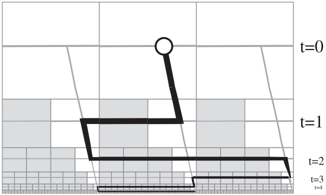

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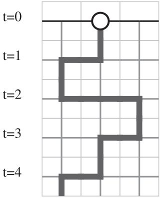

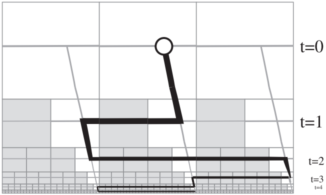

He was not able to settle whether the Domino Problem is undecidable in the hyperbolic plane and the question remained open for nearly thirty years.

Theorem (Margenstern, Kari) *The Domino Problem is undecidable in the hyperbolic plane.*

We have just seen something of Kari’s proof and are about to learn more of Margenstern’s.

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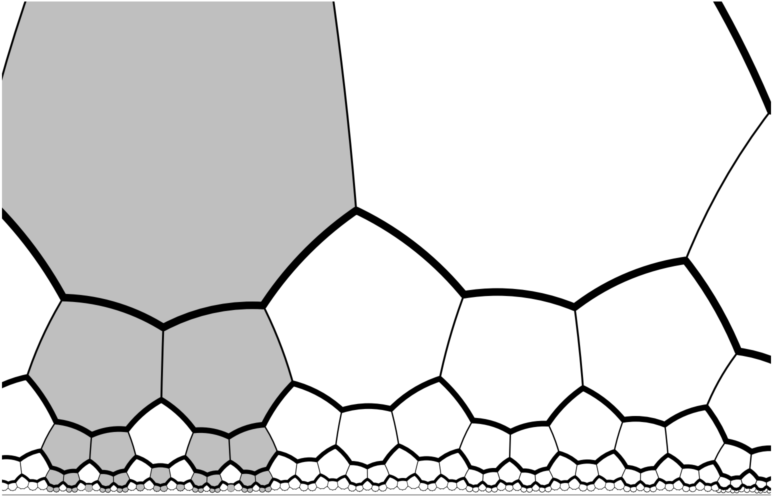
I will sketch here a simple “hierarchical" aperiodic set of tiles; these can be used as the basis for a proof very much like Berger’s, Robinson’s and Margenstern’s.

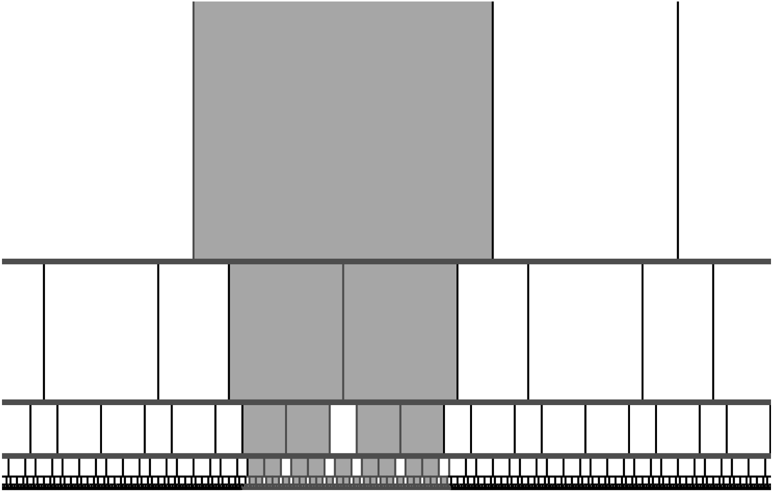
In essence, we adapt Robinson’s set of tiles to the hyperbolic plane.

But how can there be a hierarchical structure to exploit, where there are no self-similarities?

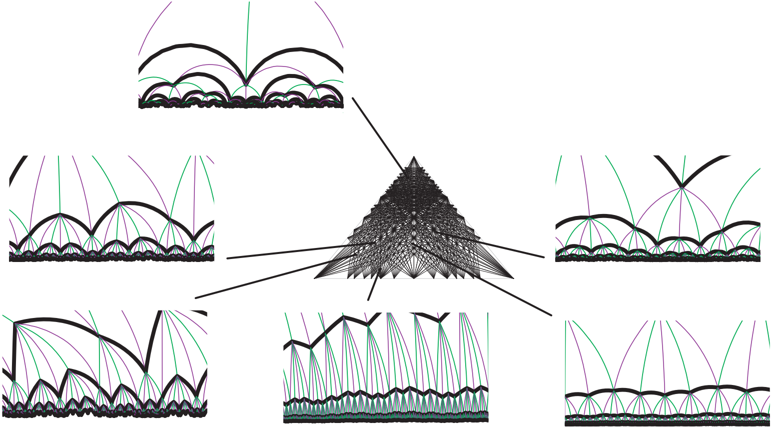
Margenstern begins with the {7,3} tiling



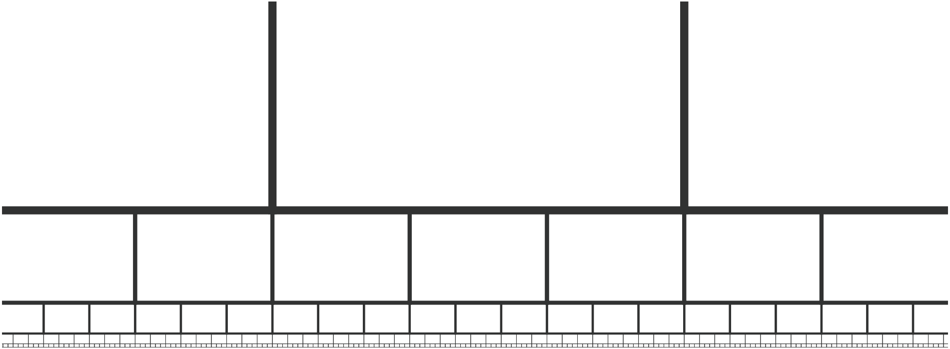
but this is already equivalent to



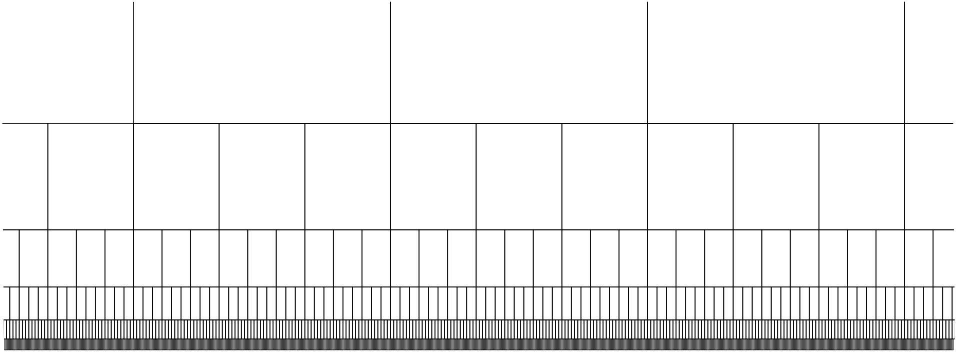
(Incidentally, this is quite a general principle, that in the hyperbolic plane, tilings are naturally viewed as geometric realizations of a kind of “regular production system", and that a given system gives rise to many, non-quasi-isometric tilings with the same combinatorial structure.)



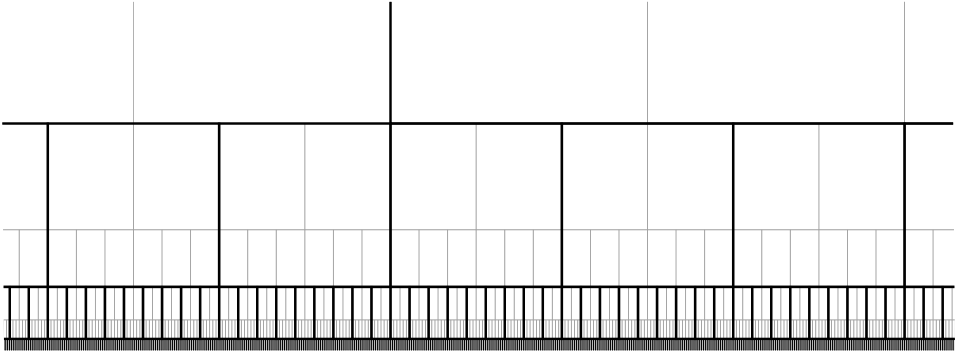
A 3-fold horocyclic tiling:



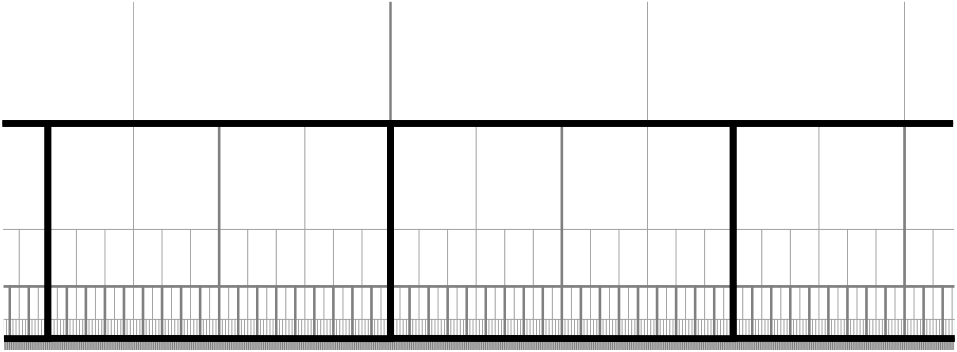
A 3-fold horocyclic tiling, in an unusual model for clarity:



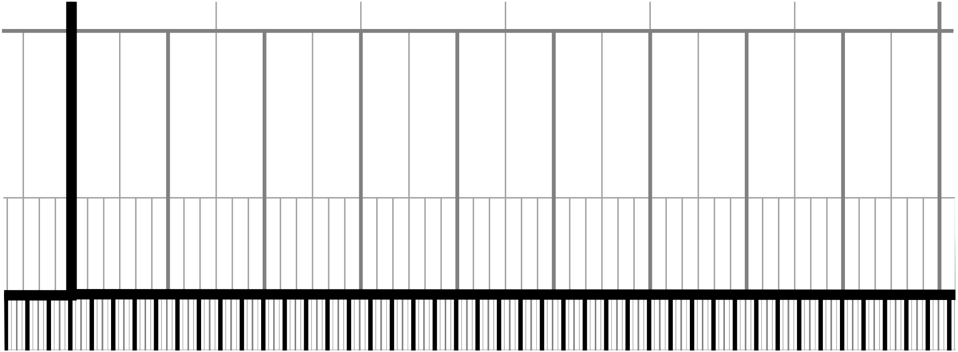
A 9-fold horocyclic tiling:



An 81-fold horocyclic tiling:



An 81-fold horocyclic tiling, zooming in



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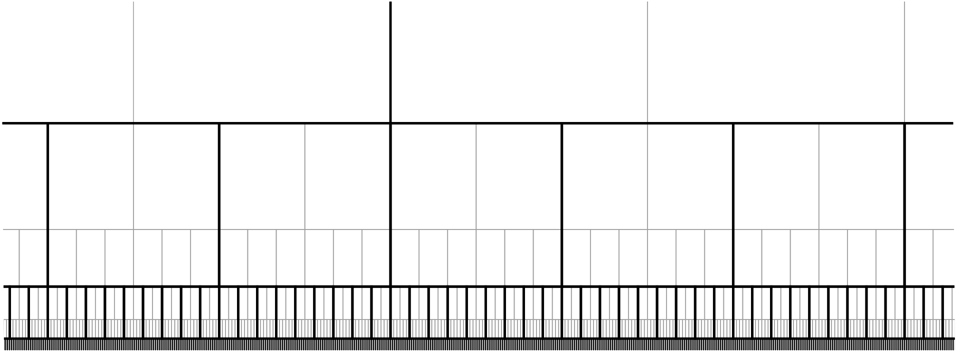
(proof)

Corollary *Any set of tiles that does admit tilings of H^2, but only tilings exhibiting this structure is* strongly aperiodic.

Now having seen this brilliantly simple structure, those familiar with the construction of matching rules will find the rest of this talk predictable.



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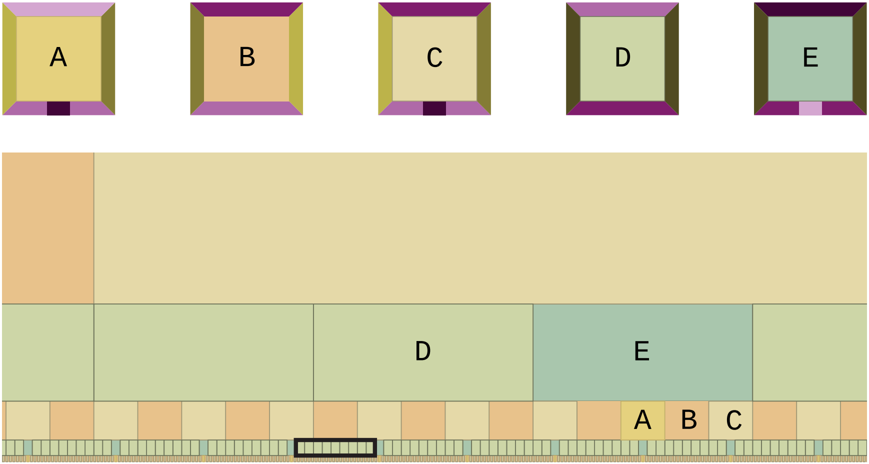
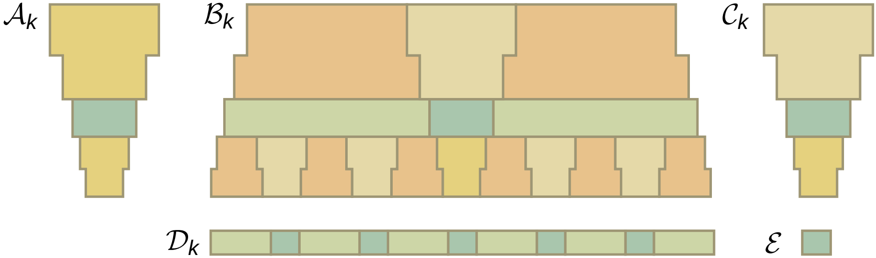
Once we have this underlying strongly aperiodic set of tiles, then the rest of the Berger-Robinson-Margenstern proof is relatively standardized.

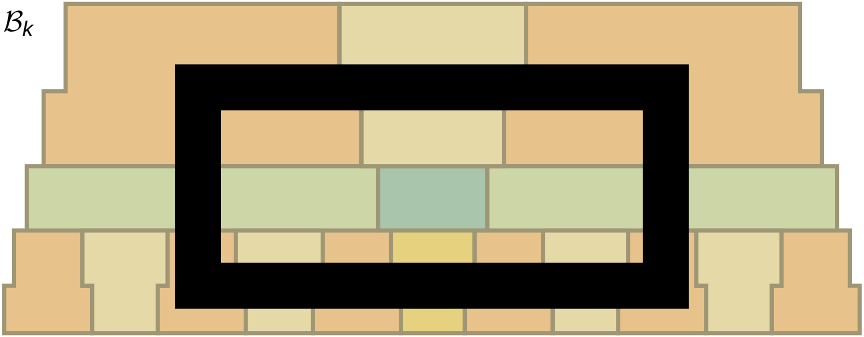
Five basic tiles:



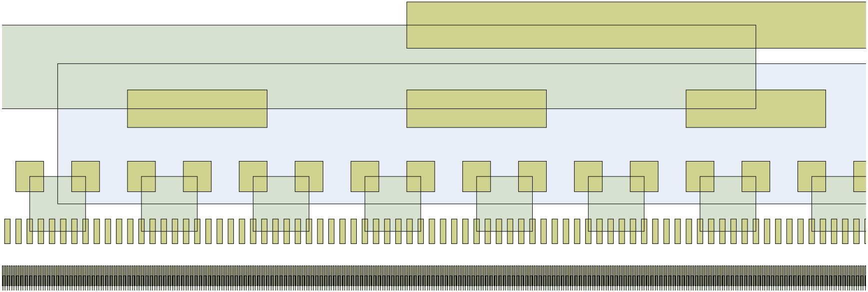
Inductively define “blocks"



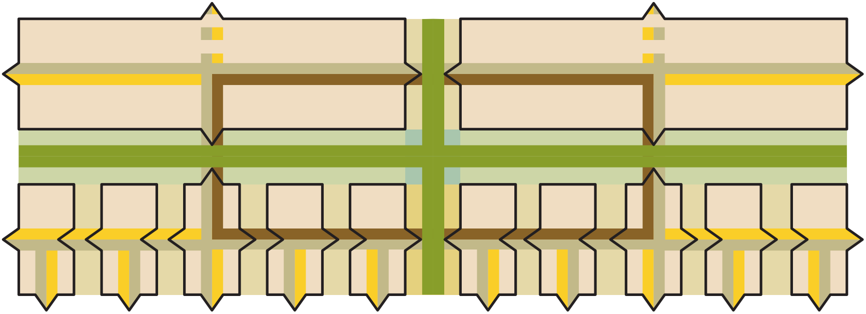
We aim to define markings that delineate a “nerve" in each $\mathcal{B}_k$:



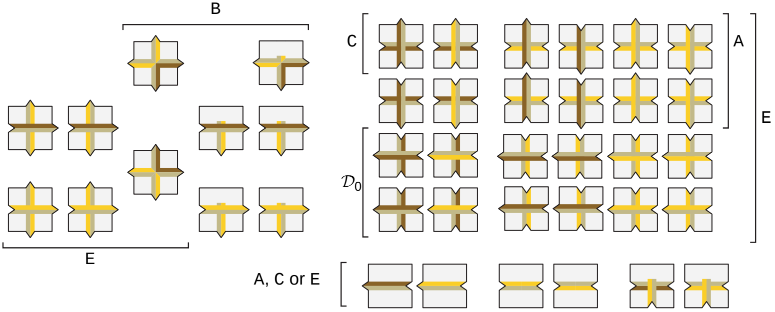
A hierarchy of nerves:



A schematic of the markings:



85 tiles:



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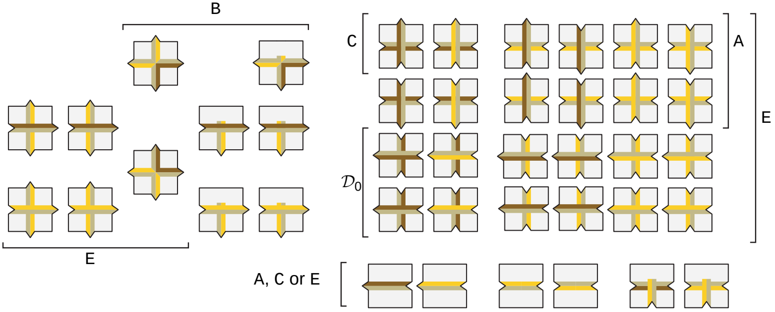
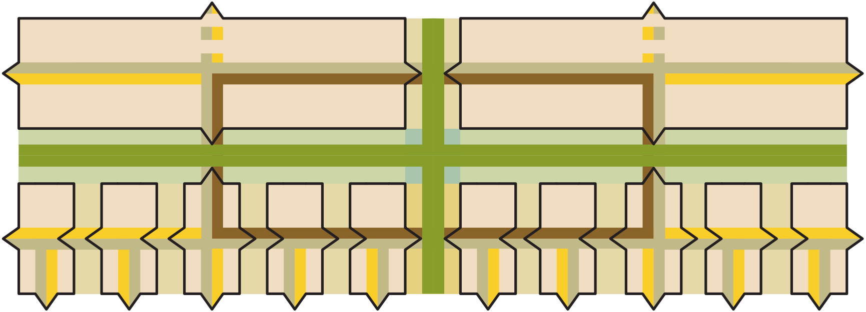
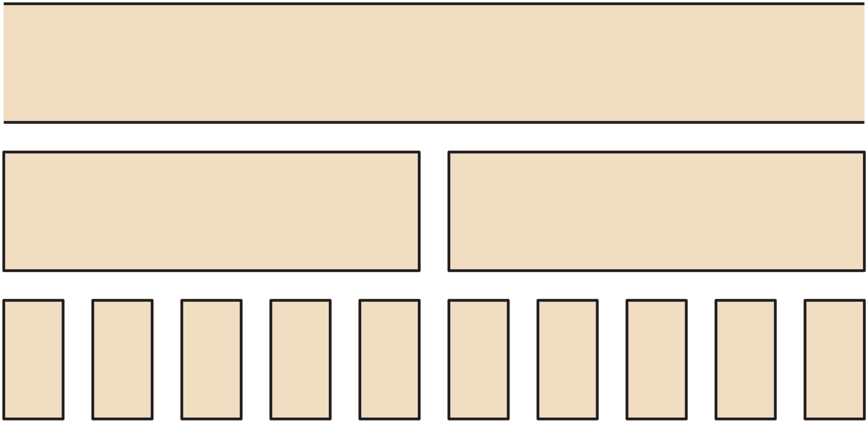


(fewer if we allow non-standard rules—such as “tip to tip")

A “well-formed marked k-level block $\mathcal{B}_k$"



The Proof: Assume every tile B lies in a unique k-level well-formed block, and these blocks are arranged in an $*$-fold horocyclic tiling. We show this is so for $k+1$.



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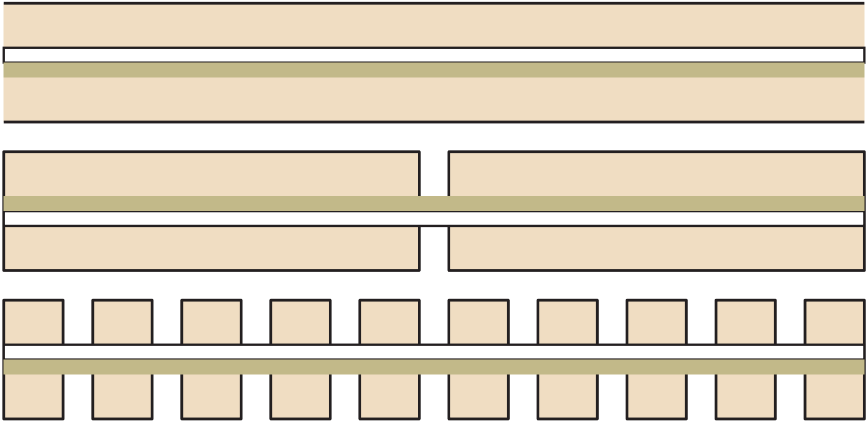
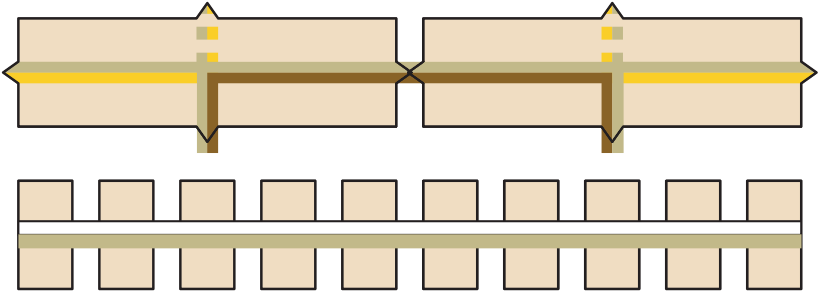
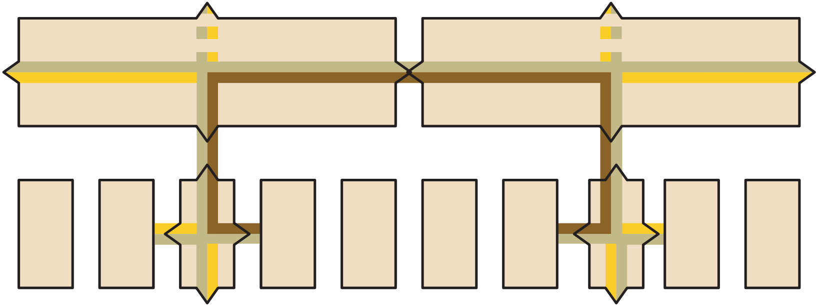
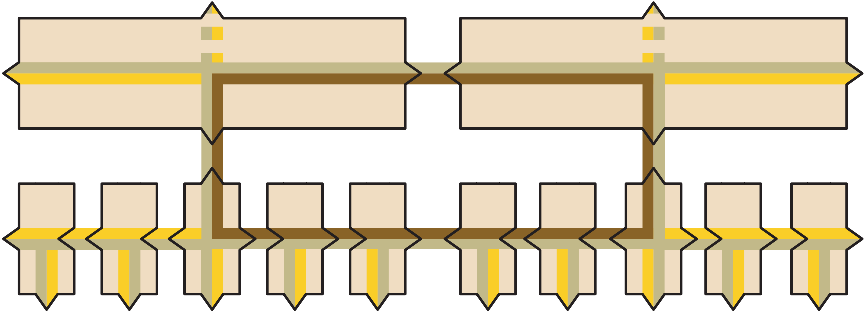
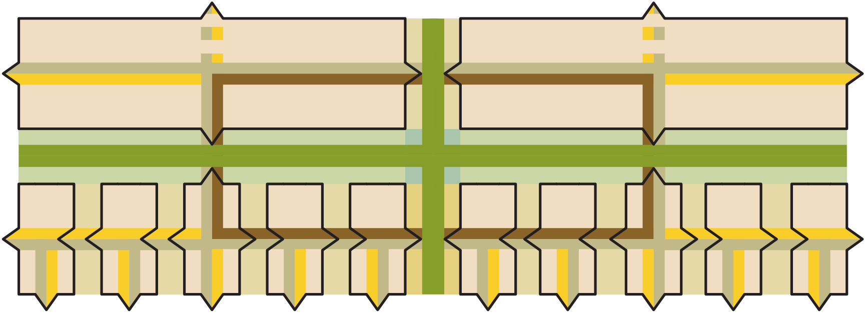
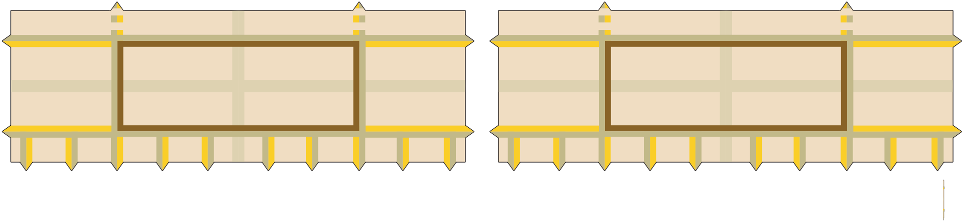




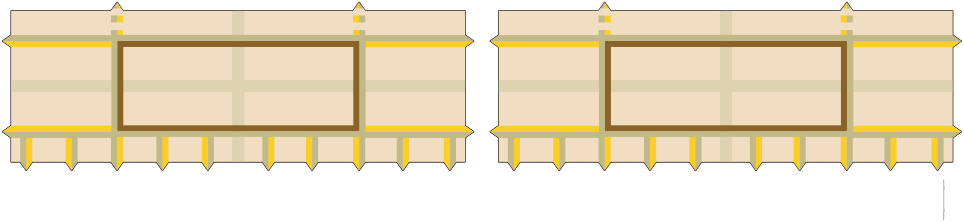






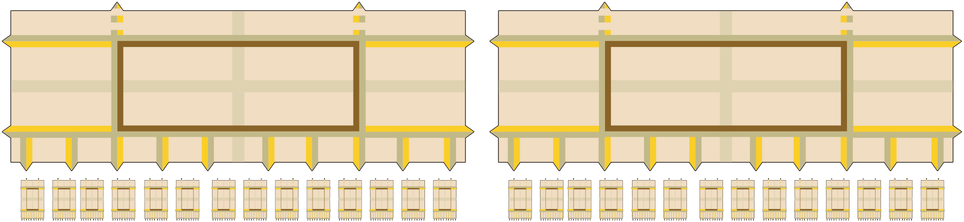


These well-formed blocks cannot overlap



These well-formed blocks cannot overlap

and can only lie in neat rows, the ends of each block directly above the ends of others.



These well-formed blocks cannot overlap

and can only lie in neat rows, the ends of each block directly above the ends of others.

QED: This set of 85 tiles is strongly aperiodic, forming only “hierarchical" tilings in the hyperbolic plane.