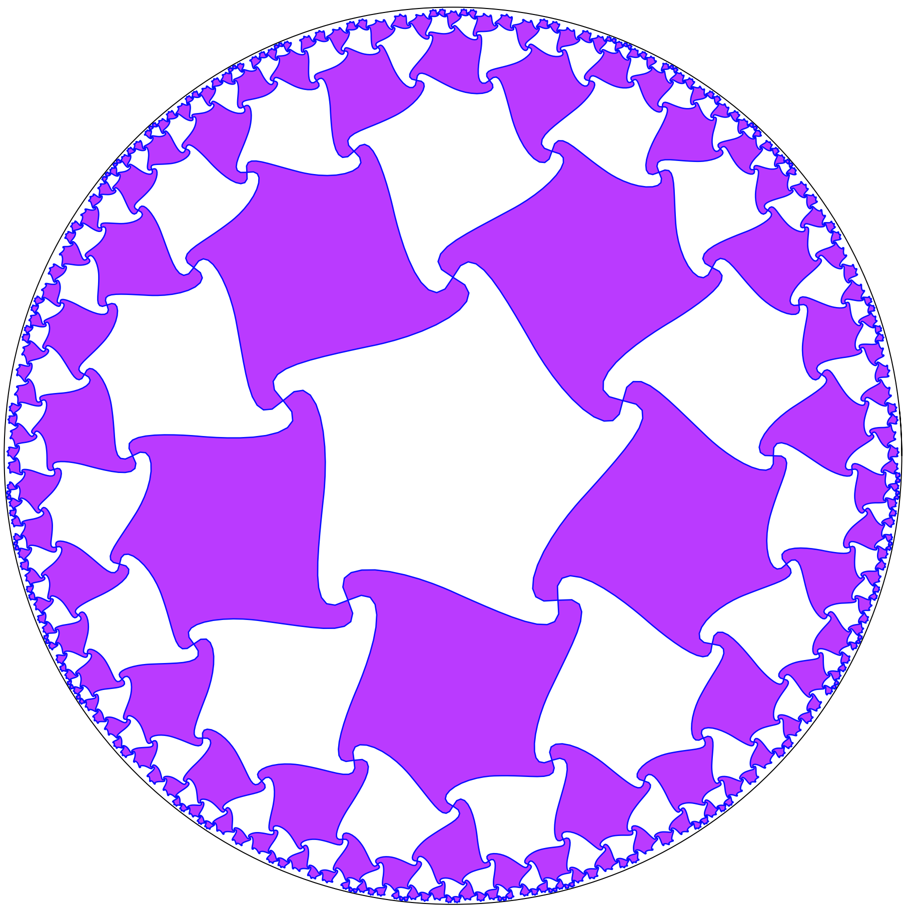
How to Create Problems

(in Tiling)



In any given fixed setting, say polygonal tiles in the hyperbolic plane, we can ask a variety of questions, such as:

"Is there an aperiodic set of tiles?"

In this talk, we'll look at an interlocked web of such questions and discuss a variety of conjectures and results, both old and new.

In general, a particular setting is given by

• a metric space X,
• a group of isometries $\mathcal{G}$ acting on X to move tiles about,
• and a description R of allowed tiles, local matching rules, etc.

A setting is "nice" if we can enumerate, for each finite set of tiles (each protoset), all finite configurations.

In this talk we'll focus mostly on these two nice settings:

$\mathbf{H}^2$, tiled by polygonal tiles meeting vertex-to-vertex.

$\mathbf{E}^n$, tiled by copies of a single tile.

But the questions we ask can be considered in any setting, and the connections between the questions hold in any nice setting.

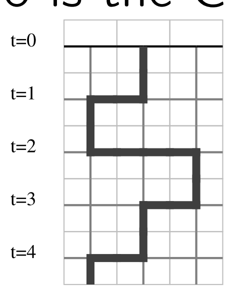
The Completion and Domino Problems

In a fixed, specific setting $(X, \mathcal{G}, R)$, we can ask the following:

Question 1. *Is there an algorithm that decides, given a set of tiles T and a starting configuration C, whether C can be extended into a tiling by C? That is, is the "Completion Problem" decidable?*

Question 2. *Is there an algorithm that, upon being given a set of prototiles T decides whether or not there is a tiling by T? That is, is the "Domino Problem" decidable?*

In 1961 Wang showed that the Completion Problem, for square tiles with colored edges moved by translations only, in $\mathbf{E}^2$ ("Wang tiles"), is undecidable by constructing, for any Turing machine, a set of tiles T so that a certain "seed" configuration could be completed to a tiling in $\Sigma(T)$ if and only if the machine fails to halt. Since the Halting Problem is undecidable, so too is the Completion Problem.



Wang asked Question 2, and conjectured the answer was positive. In particular, he could not see how one could construct a set of tiles so that the use of the seed configuration he required could be guaranteed (note that the Domino problem *is* decidable for the tiles in the above construction).

In 1964, Berger proved that the Domino Problem was in fact undecidable for Wang tiles in $\mathbf{E}^2$. In particular, then, Berger was able to construct, for any given Turing machine, a set T of tiles such that $\Sigma(T) \neq \emptyset$ if and only if the corresponding machine fails to halt. He did this by first building a "hierarchical framework" — a hierarchy of larger and larger domains forced to appear by the structure of the tiles in the protoset — on which to hang Wang's construction.

By and large, if we regard other classes of protosets in $\mathbf{E}^2$ or in higher dimensonal Euclidean spaces, these results can be extended.

However, the answer to either of these questions is unknown if we ask that T contain only a single prototile. This conjecture has surely occured to many people:

Conjecture *In the Euclidean plane, among polygonal monohedral tilings, the Completion problem and the Domino problem are decidable.*

This conjecture remains open even for "polyominoes" — tiles assembled from squares.

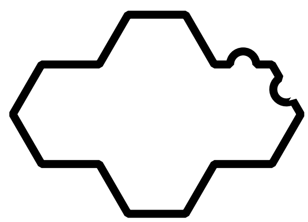
But we apply:

Conway's Presumption: *If a lot can happen, everything will happen.*

In other words, in a given combinatorial setting, if there is enough complexity that all kinds of mysterious things occur, unless you have a *really good* reason to thing otherwise, its best to bet that the setting can emulate computation, and so gives rise to undecidable questions.

Already with monohedral tilings, all kinds of mysterious examples have been found, that support an application of Conway's conjecture.

For example, is it possible to tile the plane with copies of this tile?



Naively trying things out, one finds it is possible to tile a pretty large region, but then, mysteriously, no more. Examples like this abound. Applying Conway's Presumption, we have a counter-conjecture:

Conjecture *In the Euclidean plane, among polygonal monohedral tilings, the Completion problem and the Domino problem are not decidable.*

As we get to other related questions, we'll see more support of the use of Conway's Presumption.

Let P be a set of tiles, in some setting X. One of three things is true:

• *There is no tiling of X by copies of the tiles in P.*

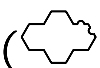
• *There is a periodic tiling (a tiling with a co-compact symmetry) of X by copies of the tiles in P.*

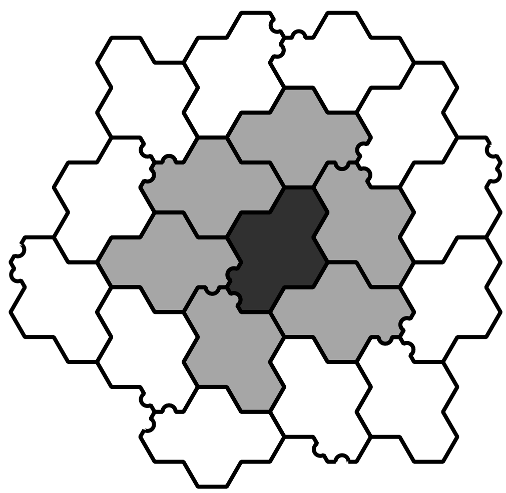
• *There is a tiling by copies of the tiles in P, but there is no periodic tiling.*

We'll look at each of these cases in turn and see how the relate to our questions.

Heesch Number

Suppose P is a set of tiles that cannot tile X. Up to certain "niceness" conditions on our setting, this implies that there is a maximum size disk which can be tiled by P. We can measure the size of this disk by counting the maximum number of *coronas* that one can form. This is the Heesch Number $H(P)$ of P. For example,

$H($$) = 2$



Let f be any function $f : \{P\} \to \mathbf{N}$ (for example, f may just give the number of tiles in the prototset, or might enumerate the possible protosets).

Let $H(n) := \max_{P \in f^{-1}(n)} H(P)$
($H(4)$, for example, might be the highest possible Heesch number among sets of 4 tiles)

Then if the domino problem is undecidable in X, there is no computable function that bounds $H(n)$.

If the domino problem is undecidable in X, there is no computable function that bounds $H(n)$!

In particular, for example, there is no computable bounding function on the Heesch number for Wang tiles. This is truly incredible if one recalls that there are computable functions such as

$$g(1) := 1, \quad g(n) := n \underbrace{!\ldots!}_{g(n-1)}$$

Question 3. *In a given "nice" setting X, is there a computable bounding function on $H(n)$?*

For monotiles, all this is a classical question posed by Heesch:

Question 4. *(Heesch) In the Euclidean plane, are there monotiles with arbitrarily high Heesch number?*

At the moment, an infinite family of monotiles with Heesch number 5 is known (Mann, 2000), but Heesch's question remains completely open. Many of the examples that are known are quite mysterious, and so applying Conway's presumption, we have

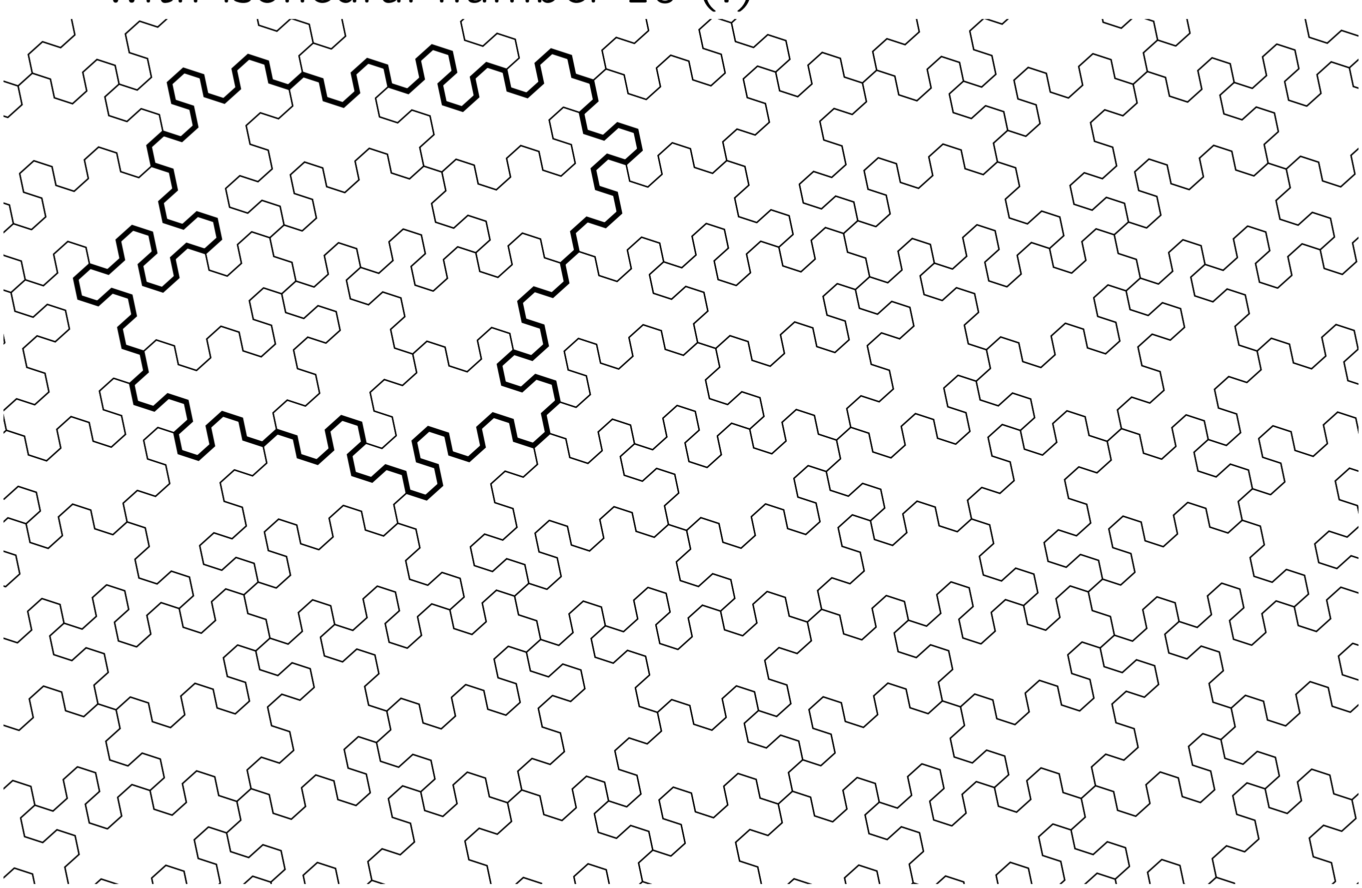
Conjecture *Yes*

Isohedral Number

Turning to the second case, we again see classical questions are tied to the Domino Problem.

Suppose P is a set of tiles, in a "nice" setting X, that admits a tiling with a co-compact symmetry. In such a tiling, the tiles lie in various orbits under the symmetry.

The *isohedral number* $I(P)$ is the smallest number of orbits possible in a tiling by P. This remarkable example is the current (Sept 2004) record holder, with isohedral number 10 (!)



Again, for any $f : \{P\} \to \mathbf{N}$ let $I(n) := \max_{P \in f^{-1}(n)} I(P)$ ($I(4)$, for example, might be the highest possible isohedral number among sets of 4 tiles)

We ask

Question 5. *In a given "nice" setting X, is there a computable bounding function on $I(n)$?*

For monotiles, again this is a classical question:

Question 6. *In the Euclidean plane, are there monotiles with arbitrarily high isohedral number?*

Very little is known; I believe, but am not sure, that examples with isohedral number 6 are the worst known. Again, Conway's Presumption leads us to:

Conjecture *Yes*

The isohedral number is related to the Period Problem:

Question 7. *In a given nice setting X, is there an algorithm to decide whether a given set of tiles admits a* periodic *tiling?*

This remains open for monotiles in the plane; very recently, K. Keating and A. Vince gave a polynomial time algorithm to decide whether a given polyomino has isohedral number 1, but this sheds little light on the general problem.

If the period problem is undecidable, then there is no computable bound on $I(n)$, and the Domino Problem is undecidable as well.

We now turn to our final case: Suppose P admits tilings but no tilings with co-compact symmetry. Well, the obvious question is:

Question 8. *In a given setting X, can such a P exist?*

(Wang) If the domino problem is undecidable, then there is such a P.

In fact, Wang originally conjectured that no such set P could exist in the Euclidean plane. In 1966, R. Berger constructed such a set in the course of proving that the Domino Problem is undecidable in that setting. These "aperiodic" sets have been a rich source of problems. . .

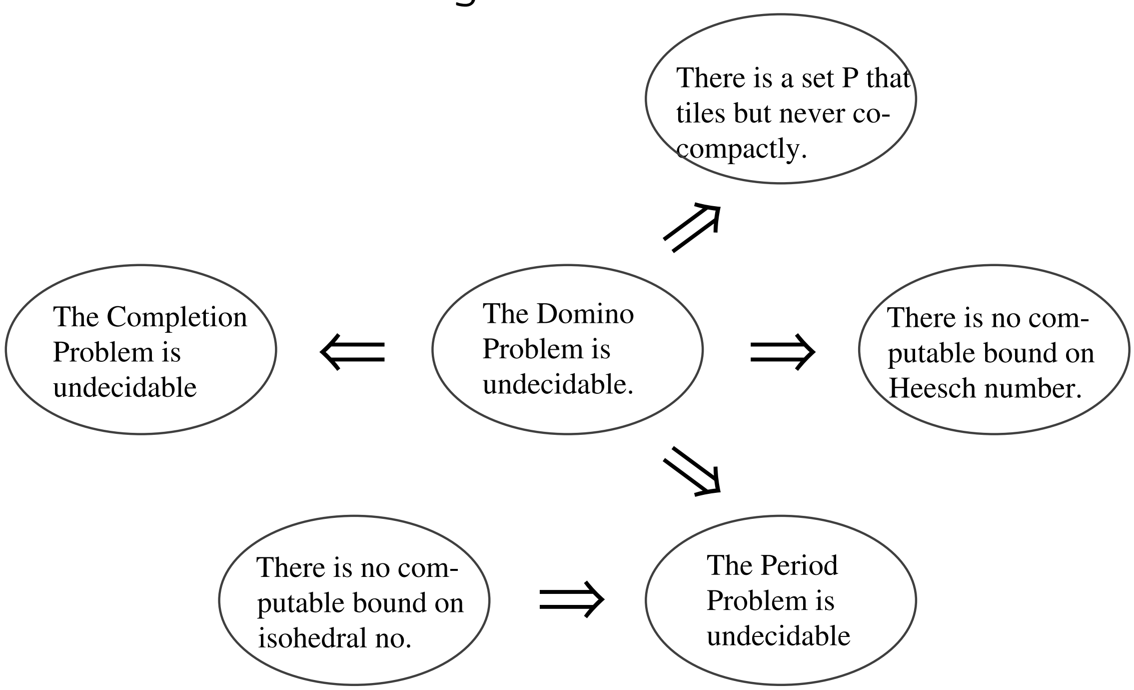
For monotiles we have:

Question 9. *Is there a monotile that admits a tiling of the plane, but admits no tiling with co-compact symmetry?*

Again Conway's presumption leads us to

Conjecture *Yes!* but to be honest, I don't believe it!

In summary, so far, we have the following implications in a nice setting X.

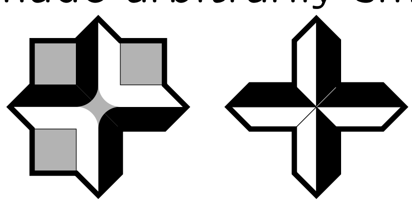


For monotiles in the plane, my brain says the answers should be *Yes* but, frankly, my heart says none of these should hold!

Incidentally, I'd like to briefly discuss a couple of near misses in the search for a monotile that tiles but not co-compactly.

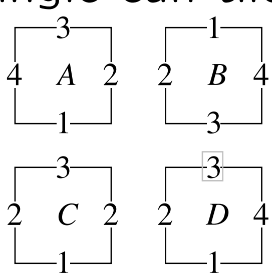
First, in $\mathbf{E}^3$, there is an example of a single tile that admits a tiling, but not with co-compact symmetry. I'll discuss this example in more detail in just a moment.

Second, there are several examples of a sets of just *two* tiles that admit tilings, but not co-compactly; the Penrose tiles are the most famous examples. Moreover there are examples in which the area of one tile can be made arbitrarily small.

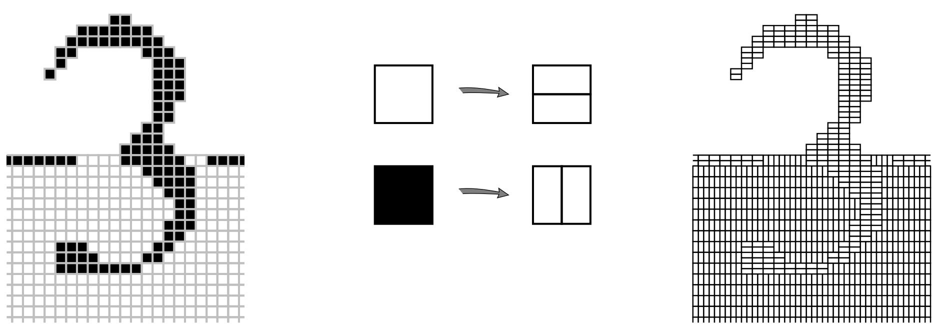


Third, one has to be very careful with the definition of the setting we're discussing. Many times, authors have looked at examples with various kinds of "matching rules"; for *this* problem we must be allowing something pretty restrictive. We certainly can't be allowing tiles with "atlas" style matching rules, because of the following, extremely stupid example:

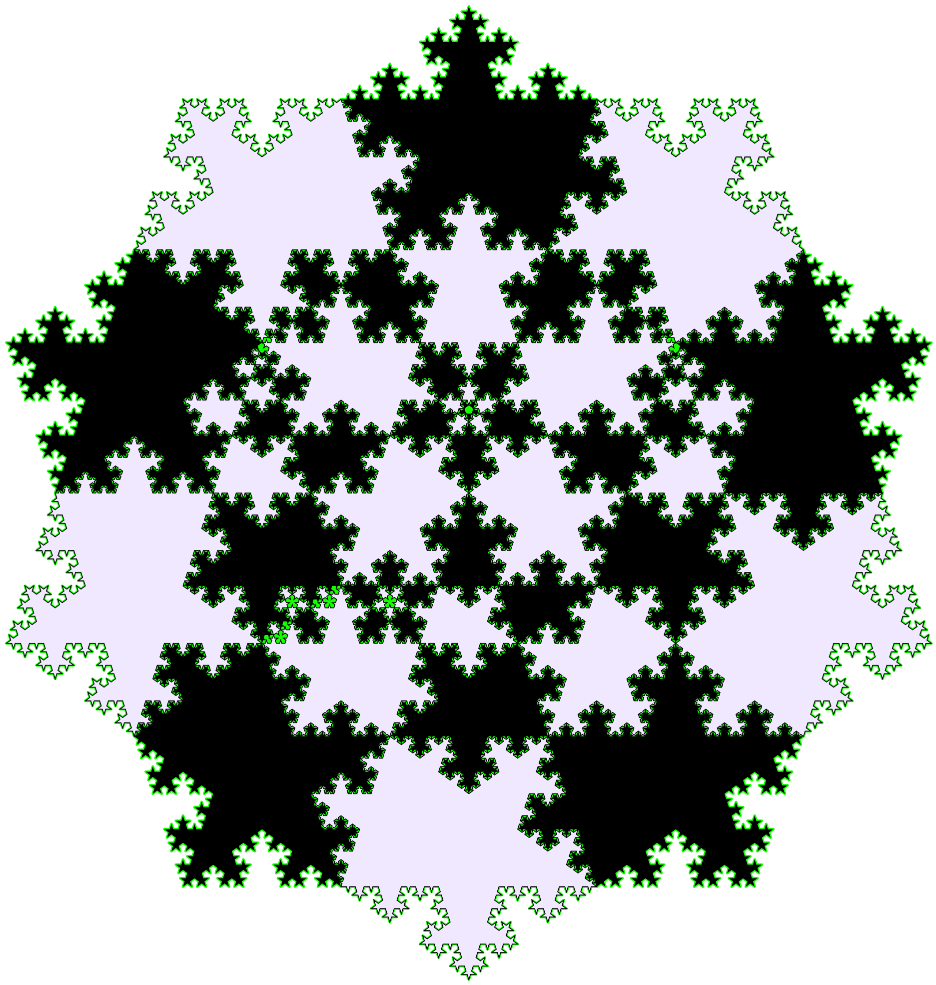
There is a set of atlas style rules that allow a 2:1 rectangle can tile but not co-compactly.



start with a set of wang-tiles with the desired property. Use these to construct an atlas for 2:1 rectangles



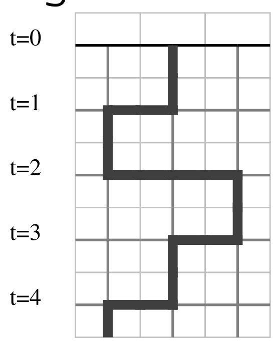
Finally, if we allow similarities, we get some fascinating examples with the desired property:

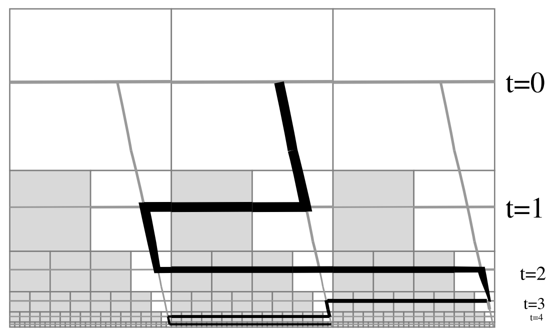


Tilings in the hyperbolic plane

I'd like to now turn to the hyperbolic plane, where some more general issues come into play.

In 1977, R.M. Robinson showed that the Completion Problem is undecidable in $\mathbf{H}^2$. Essentially he modeled Wang's original result on square tiles within a tiling of $\mathbf{H}^2$.

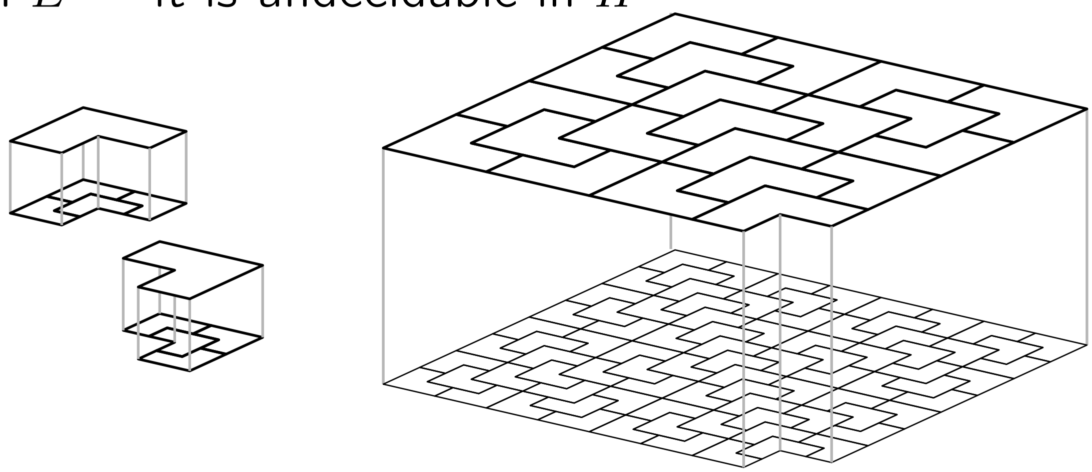




But the decidability of the Domino Problem remains open in the hyperbolic plane. Applying Conway's Presumption we have

Conjecture *In* $\mathbf{H}^2$*, among polygonal protosets, the Domino problem is undecidable*

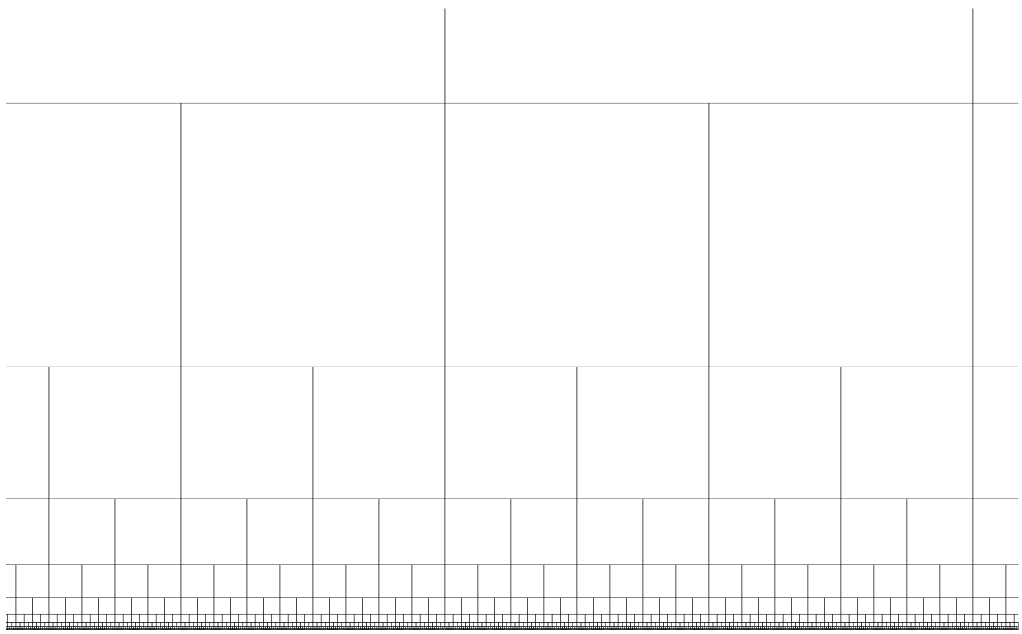
Incidentally, the Domino Problem is *not* decidable in $\mathbf{H}^n, n > 2$, by a simple trick: one can foliate H^n into parallel horospherical, $(n-1)$-dimensional Euclidean layers. For any given set T of tiles in E^{n-1}, it is trivial to construct a set of tiles in H^n that, within such horospherical sheets, mimic the behavior of the tiles in T. As the Domino Problem is undecidable in $E^{n\geq 2}$ it is undecidable in $H^{n\geq 3}$



Similarly, one can ask about bounds on isohedral number, Heesch number, and whether the Period Problem is decidable. But I'd like to focus on:

Does there exist a set of tiles that does admit a tiling but not with a co-compact symmetry?

In $\mathbf{H}^n$, the answer is yes!



However, there is something quite unsatisfying about this example. It does in fact admit a tiling that is invarient under some isometry.

In E^3, here is an example of a single tile that can admit a tiling but without a co-compact symmetry. Again though it does admit a tiling with a period. Does this tile deserve to be considered “aperiodic”

I've deliberately avoided the use of the word "aperiodic" because I wanted to make a finer distinction. This subtlety was not noticed for a long time because for tiles in the Euclidean plane, the following two definitions coincide:

A set P is **weakly aperiodic** if it admits a tiling, but no tiling with co-compact symmetry. The previous two examples are weakly aperiodic.

A set P is **strongly aperiodic** if it admits a tiling, but no tiling invarient under an infinite cyclic symmetry (a period!)

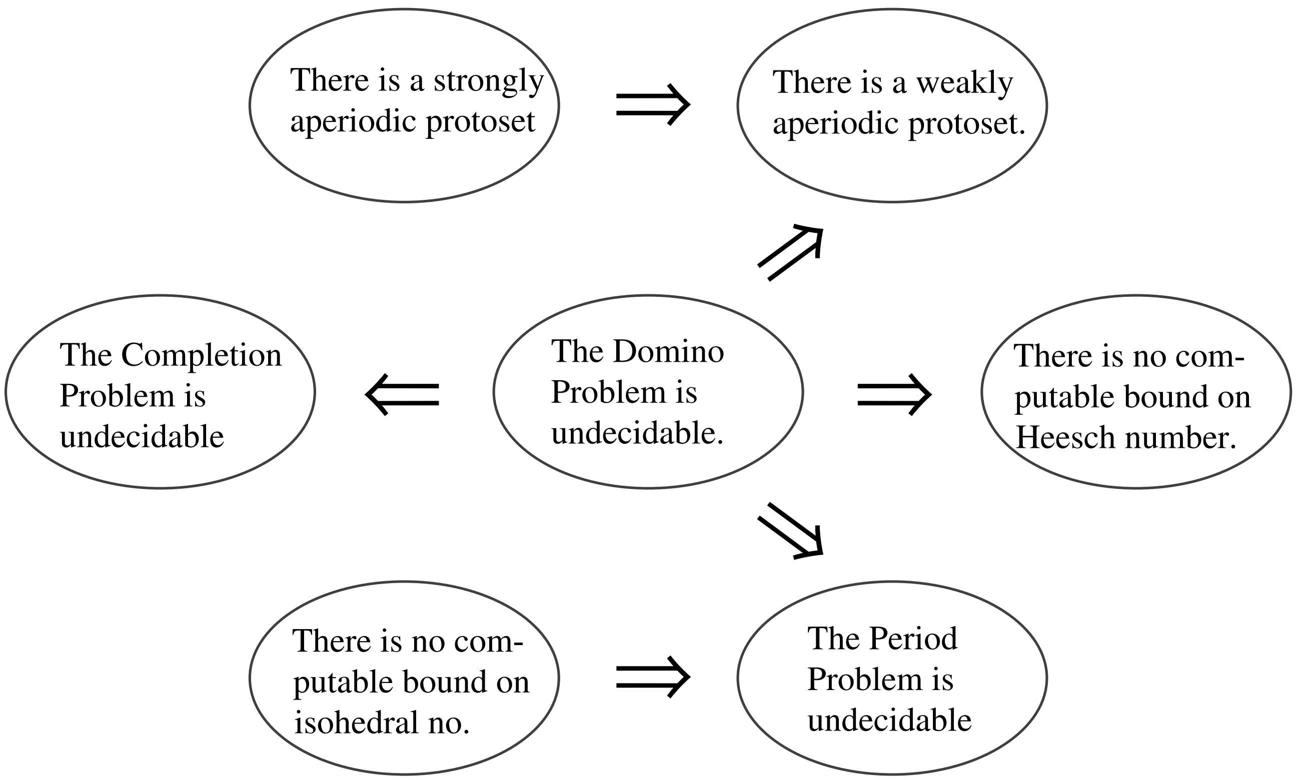
As we've seen, *weak aperiodicity* is logically linked to the Domino Problem. But somehow, *strong aperiodicity* fits more with one's sense of just what the word "aperiodicity" should mean— no periods!

And so we ask, in a given nice setting:

Question 10. *Is there a strongly aperiodic protoset?*

In particular, this remained open in the hyperbolic plane. I personally was positive no such set could exist, and I tried very hard to prove this. But eventually I realized why I couldn't succeed:

Theorem 11. *There is a strongly aperiodic protoset in the hyperbolic plane*


There is a strongly aperiodic protoset
There is a weakly aperiodic protoset.
The Completion Problem is undecidable
The Domino Problem is undecidable.
There is no computable bound on Heesch number.
There is no computable bound on isohedral no.
The Period Problem is undecidable