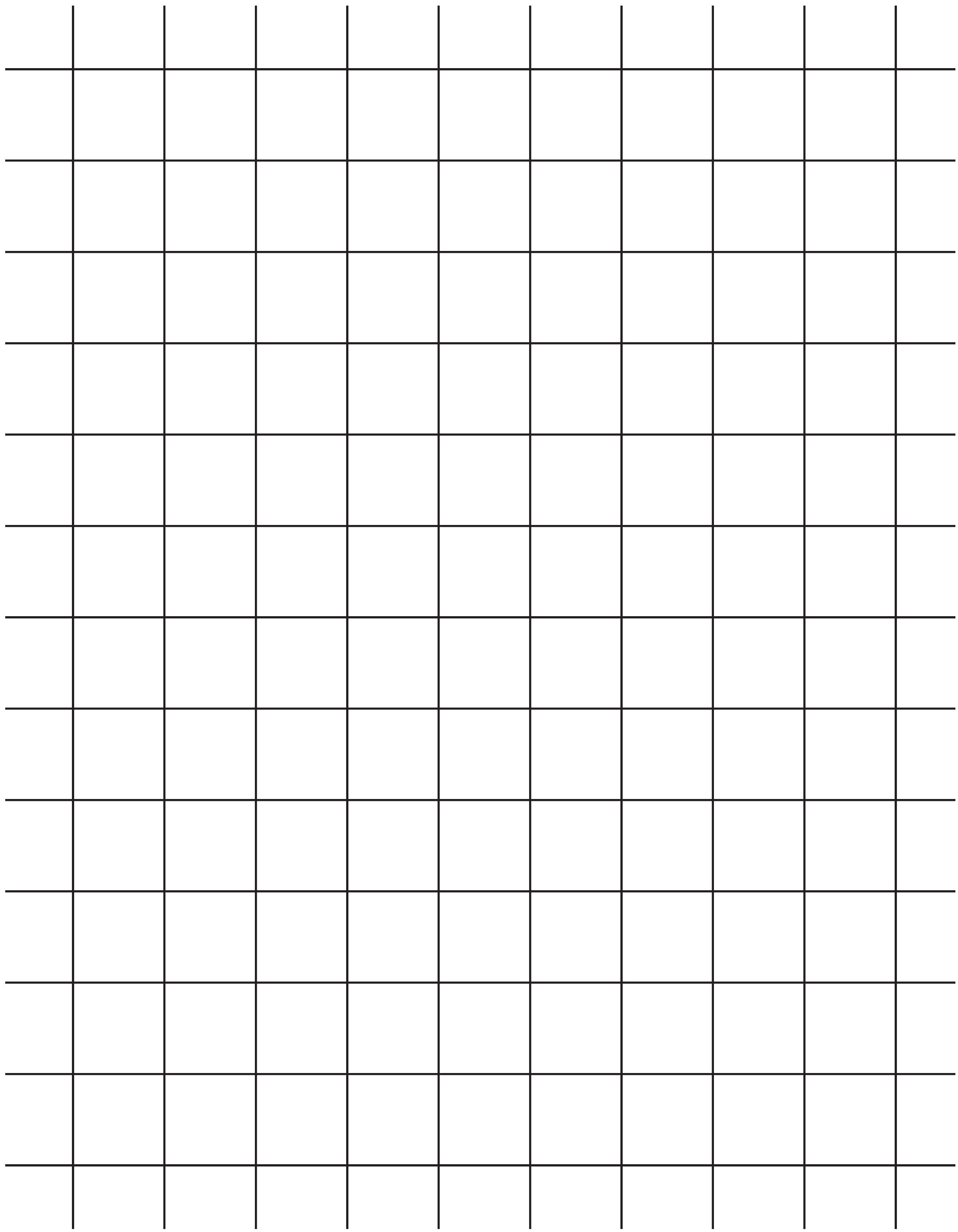
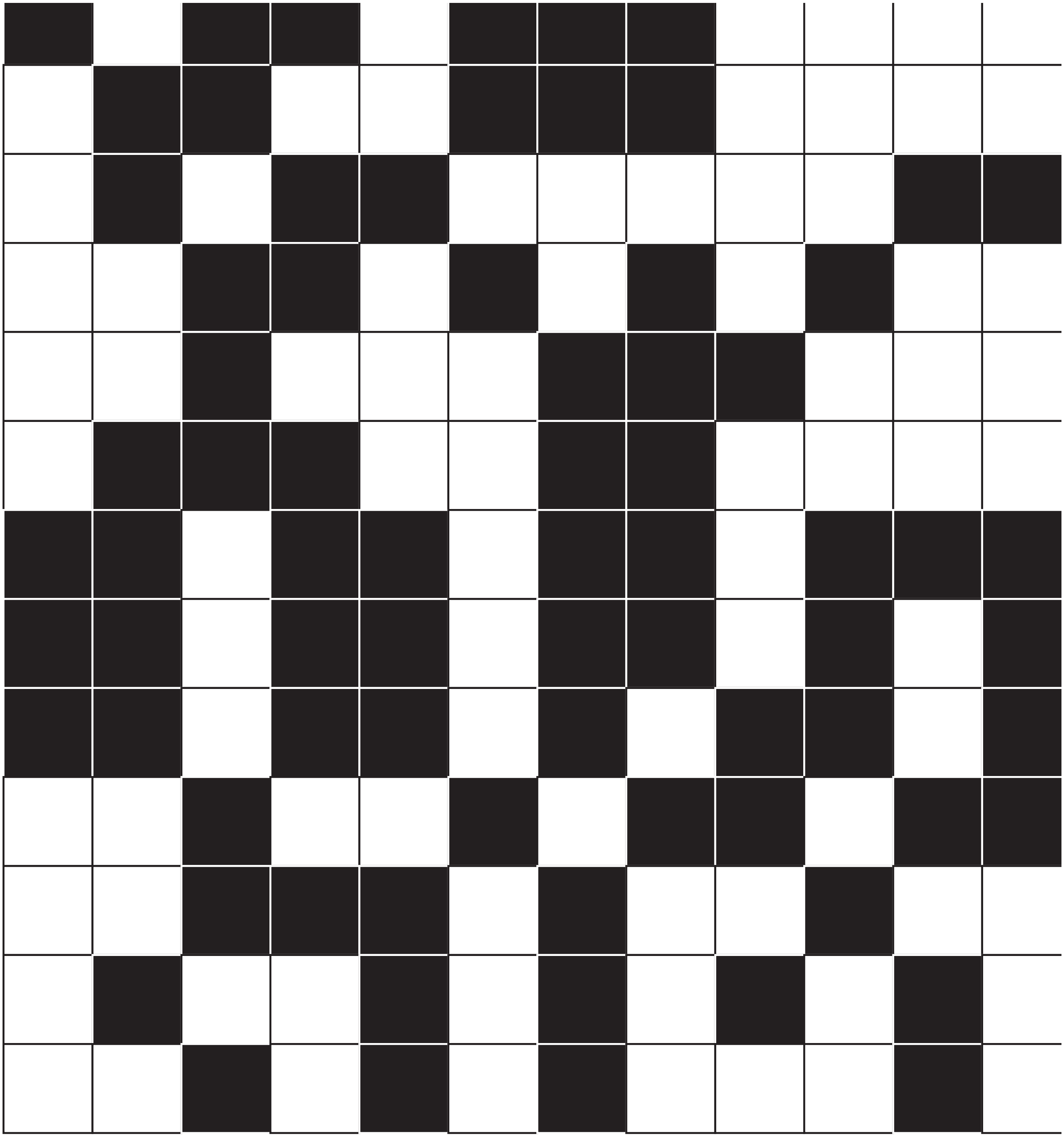
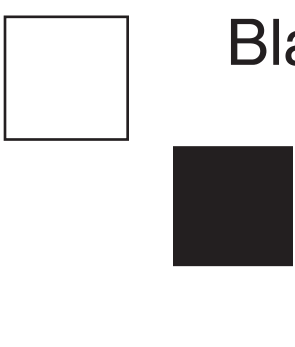
Aperiodic Hierarchical Tilings

Chaim Goodman-Strauss

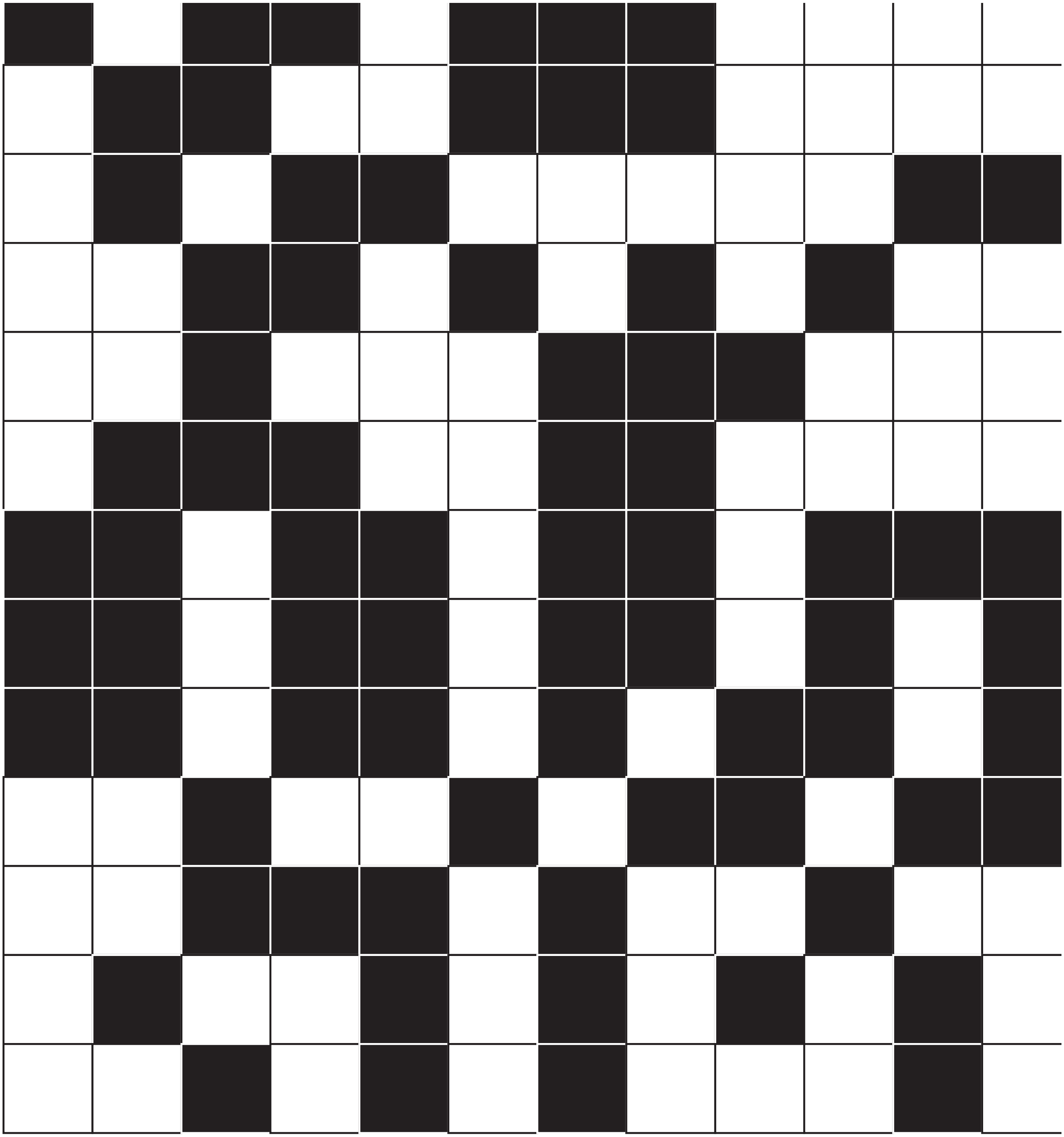
we begin with a periodic tiling:

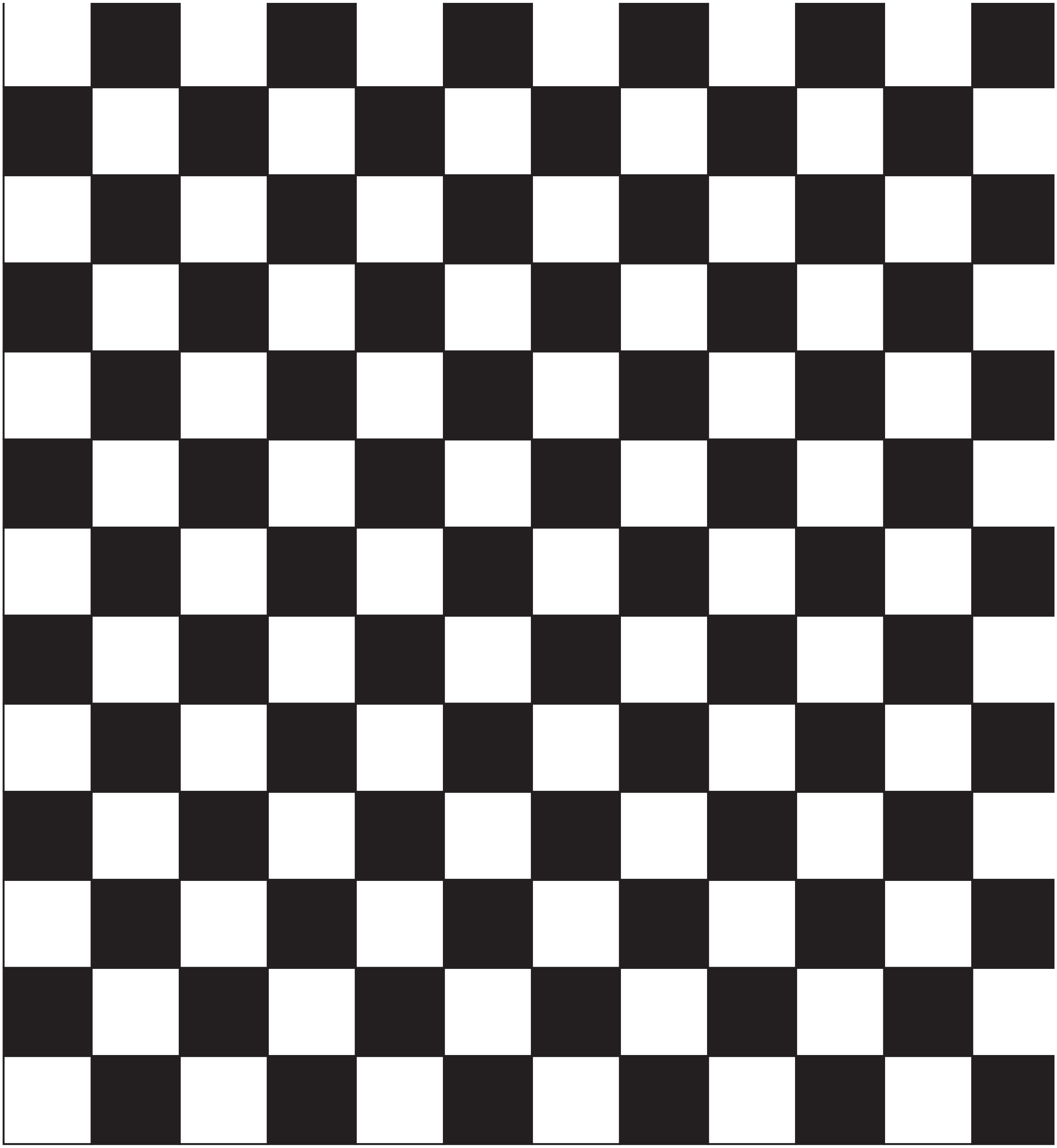




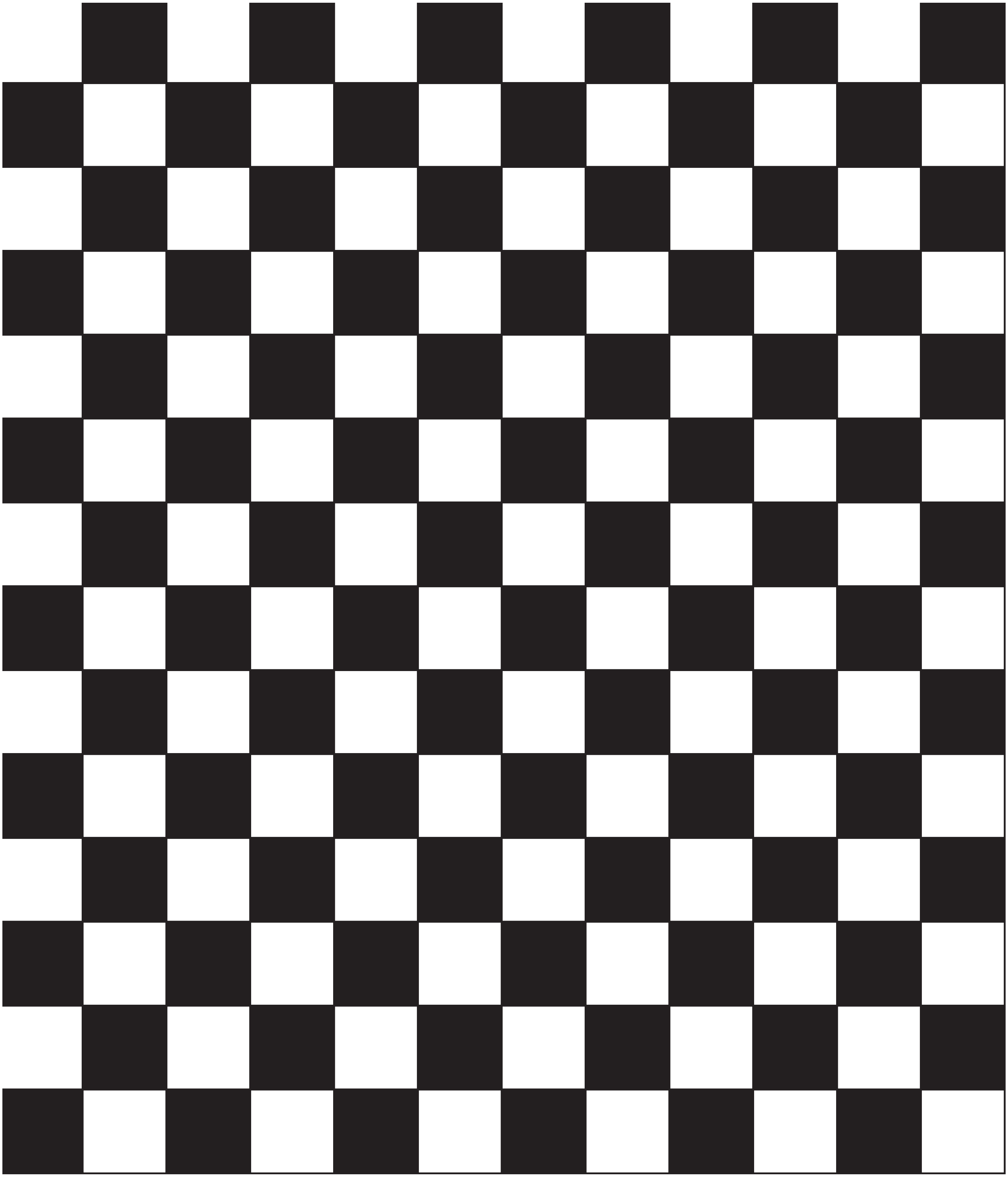


Black and white squares can tile
non-periodically





But aren't **aperiodic** since they can also tile periodically



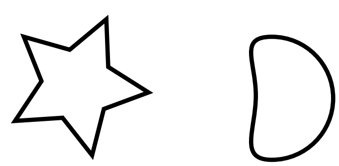
Hao Wang asked two questions in 1964:

Q1) *Is there an aperiodic set of tiles?*

That is, is there a set of tiles that can tile the plane non-periodically, but cannot tile the plane periodically?

Q2) *Is there a general way to tell if a given set of tiles can tile the plane at all?*

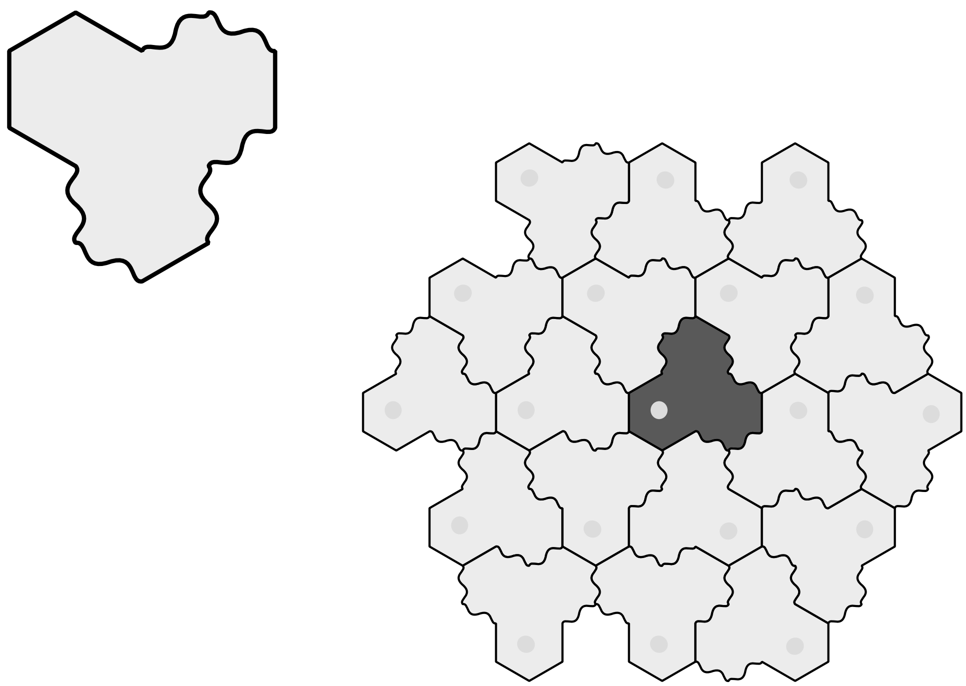
For example, these can't, but is there a *general* method to check this?



Amazingly, if there is no such general method, there must be an aperiodic set of tiles.

Why would it be difficult to tell if a set of tiles can tile the plane?

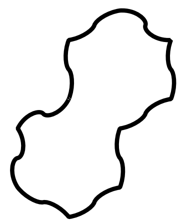
Consider this example:

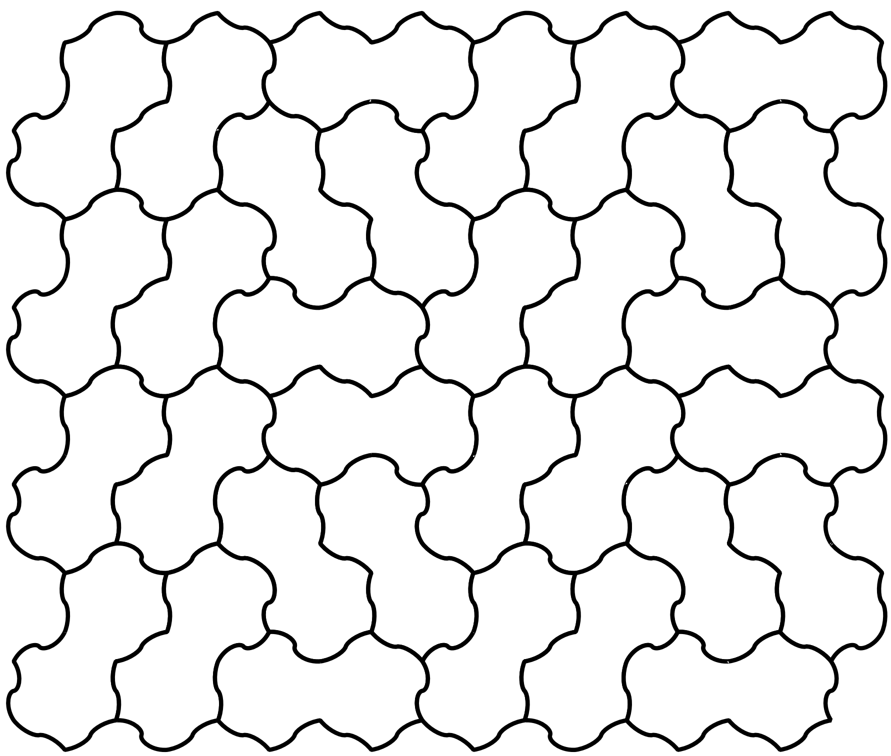


This tile can tile this much and no more; Wang's second question can be interpreted as: *Can arbitrarily strange things happen?*

Why would it be difficult to tell if a set of tiles can tile the plane?

Consider this example:





This tile can tile only periodically, but each period has at least eight tiles! Wang's second question can be interpreted as: *Can arbitrarily strange things happen?*

Wang's work pointed out an amazing connection between

HOW TILES CAN FIT TOGETHER
and
WHAT CAN BE COMPUTED

In particular, he showed how an arbitrary program can be encoded as a set of tiles.

In 1964, Wang's student, Robert Berger, showed that in fact:

There is no general method to tell whether a given set of tiles can be used to form a tiling!

Consequently, *there exists an aperiodic set of tiles!*

And indeed he gave such a set, though it had over 20,000 different tiles!

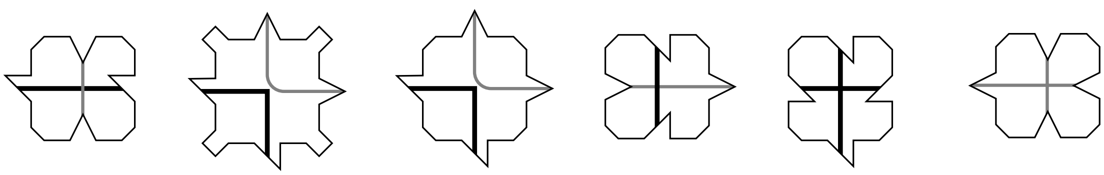
Incidentally, it follows that there are true but unprovable statements of the form:

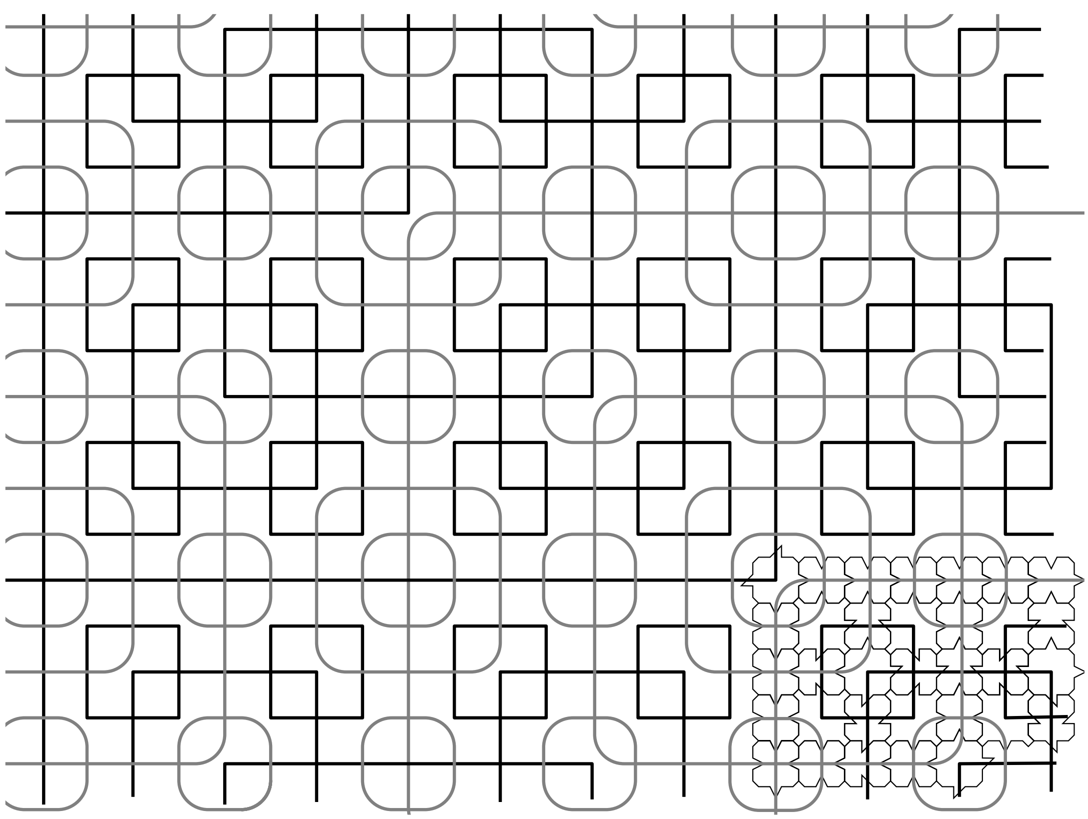
"*such and such a set of tiles can form a tiling.*"

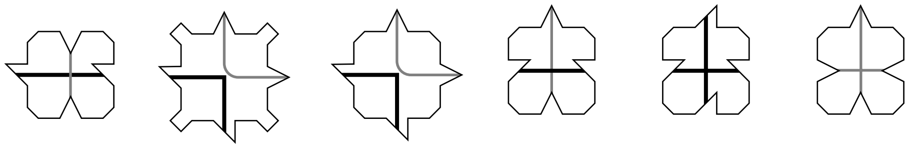
This is closely related to Gödel's Theorem, one of the deepest threads in 20th century mathematics.

Over the years, many simpler examples of apeiodic tilings wer found. Raphael Robinson gave the first really small set (and a simpler version of Berger's proof) in a lovely 1971 paper.

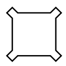
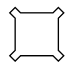
Robinson's small aperiodic set:



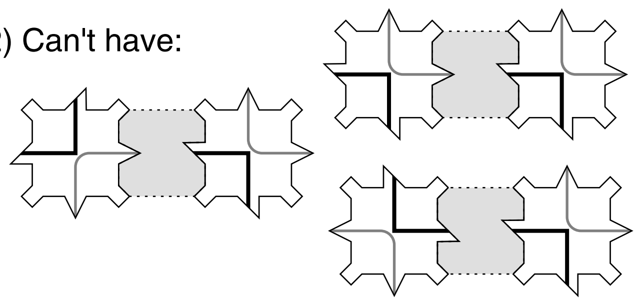




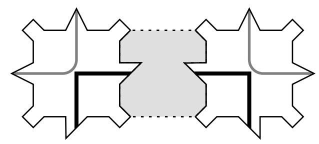
Thm: *The Robinson tiles are aperiodic. That is, no tiling with the Robinson tiles is invariant under any translation.*

1) Every tile is either a  or incident to 

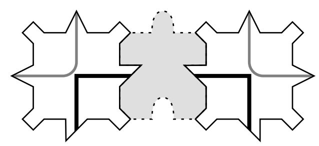
2) Can't have:

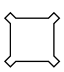


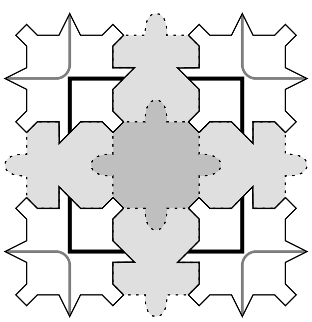
Only:



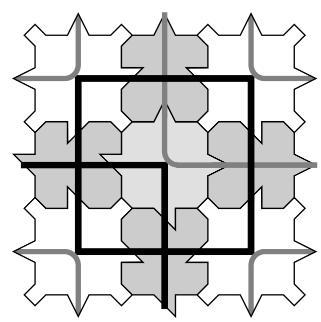
Hence:

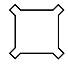


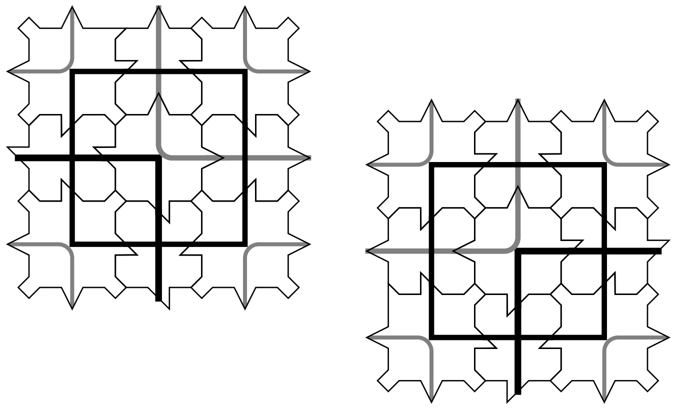
3) So each  is part of:



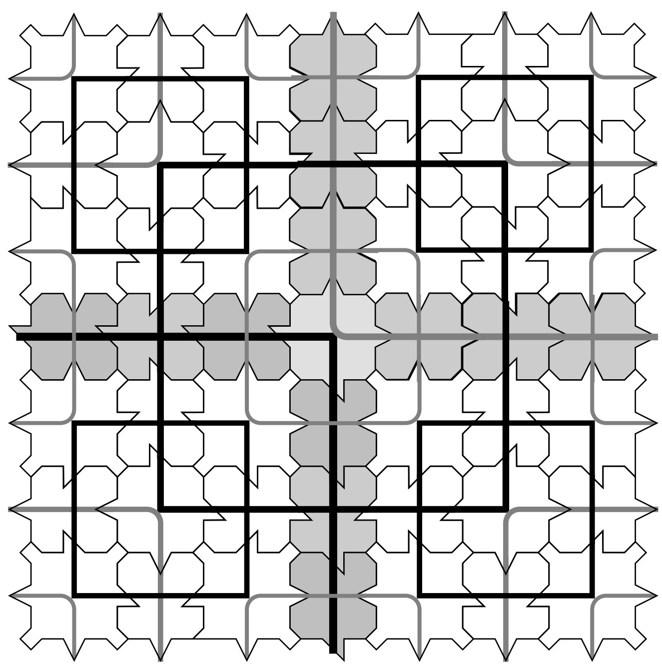
Hence, up to rotation, every tile is in or next to:



4) These 3x3 blocks act like large  's



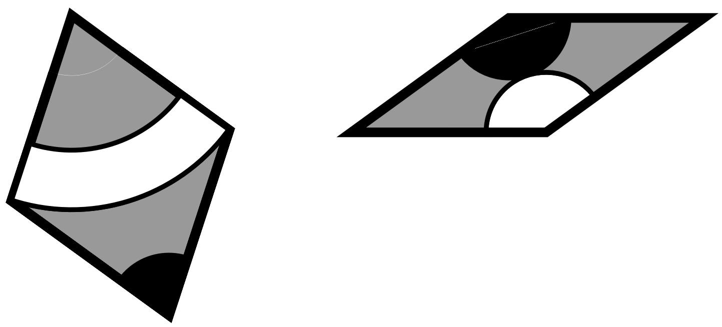
& up to rotation,
every tile is in or next to
a 7x7 block:

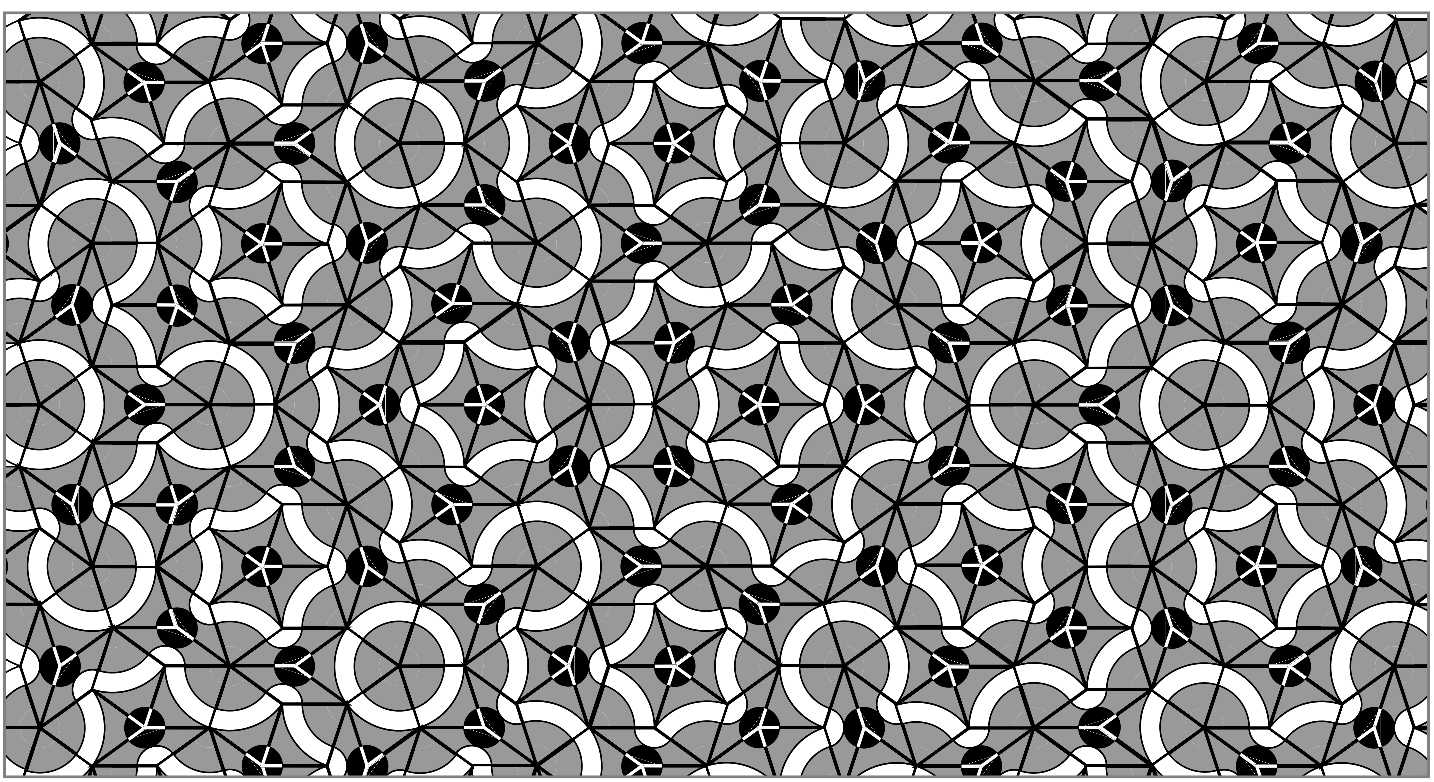


& up to rotation,
every tile is in or next to
a 15x15 block, a 31x31 block, etc...

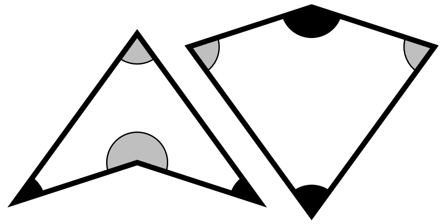
Consider a tiling by the Robinson tiles. Any translation has a finite magnitude and will translate some giant block onto itself. But this will not leave the tiling invariant. Hence every tiling by the Robinson tiles is non-periodic and the tiles themselves are aperiodic.

In 1972 Roger Penrose gave an aperiodic set of just two tiles, which was later modified by Robert Amman and John Conway. This became the most famous example:



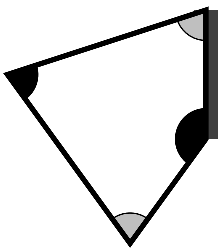


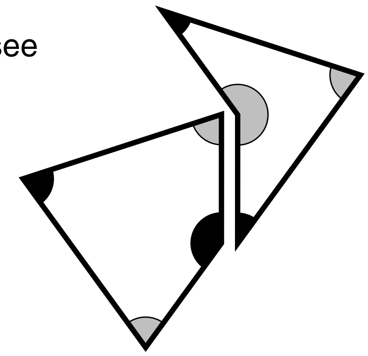
Theorem: The Penrose Kite and Dart are aperiodic tiles.



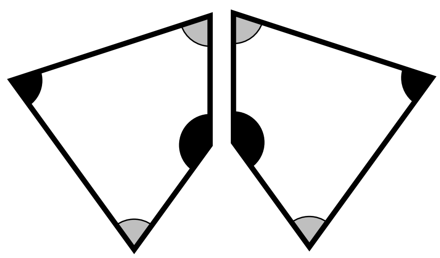
1) In any tiling with the tiles, both tiles must appear.

2) Along the short edge of a kite can only see

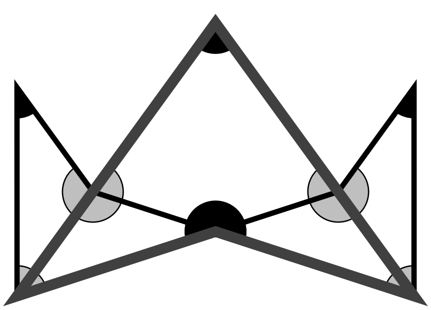


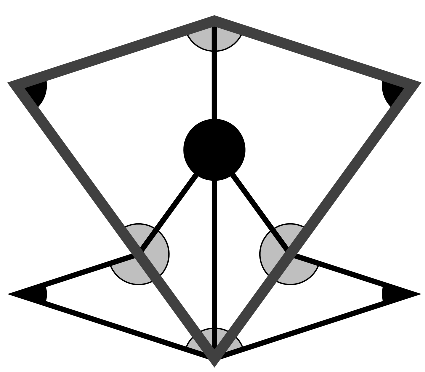


or

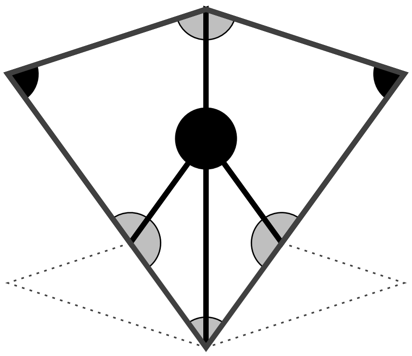


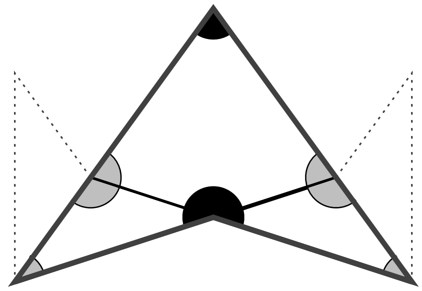
hence, every tile is in a large kite or a large dart:

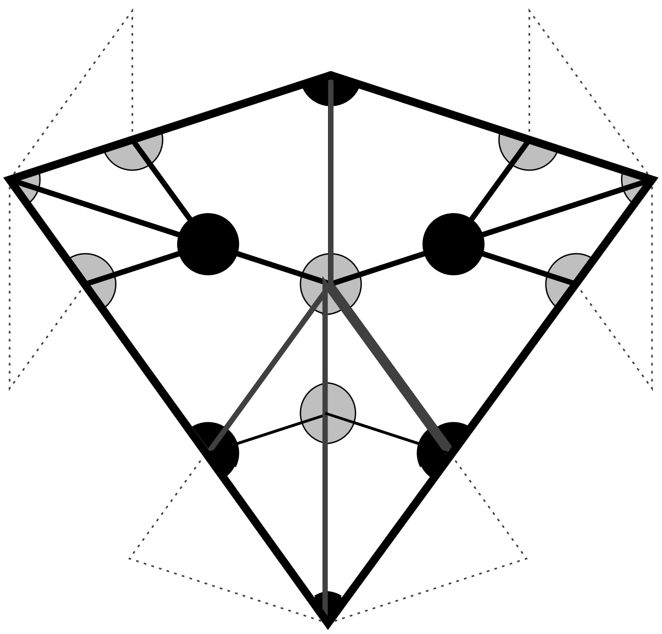


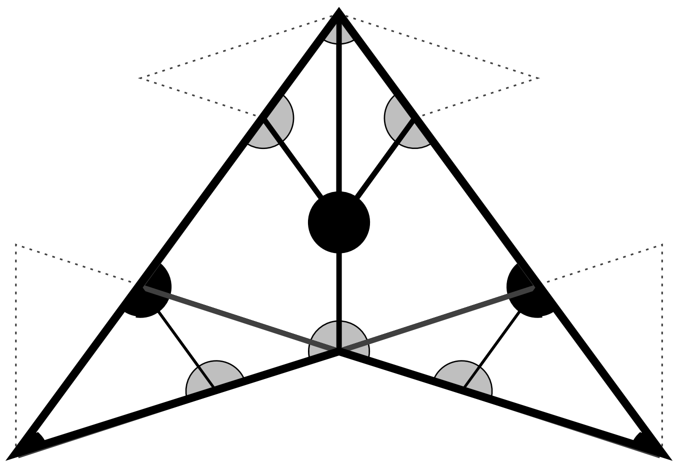


Etc:





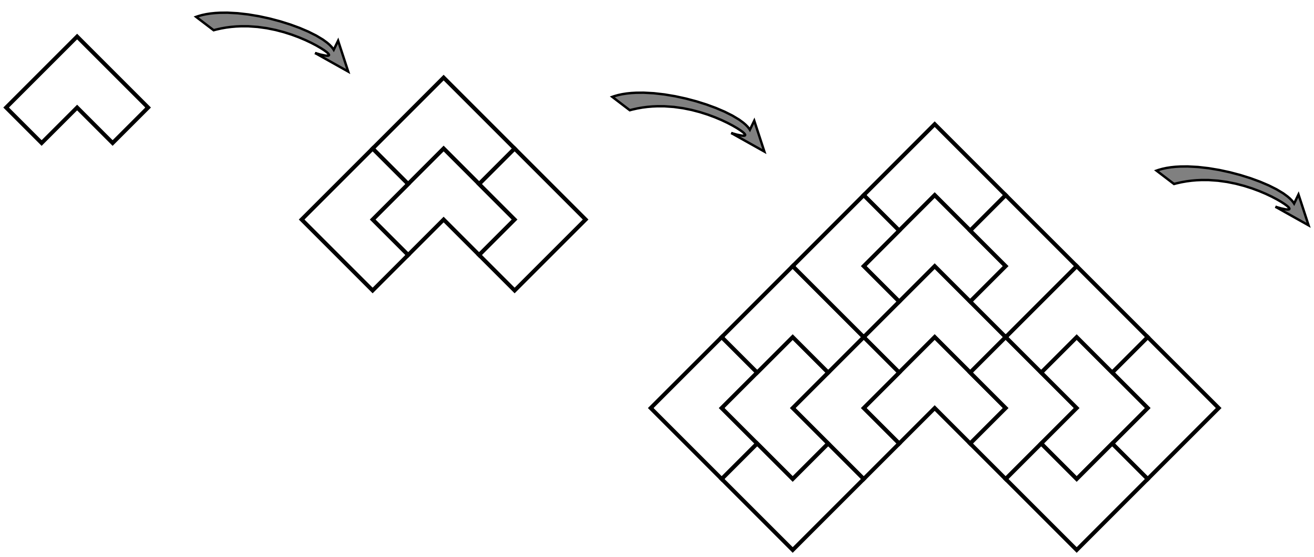


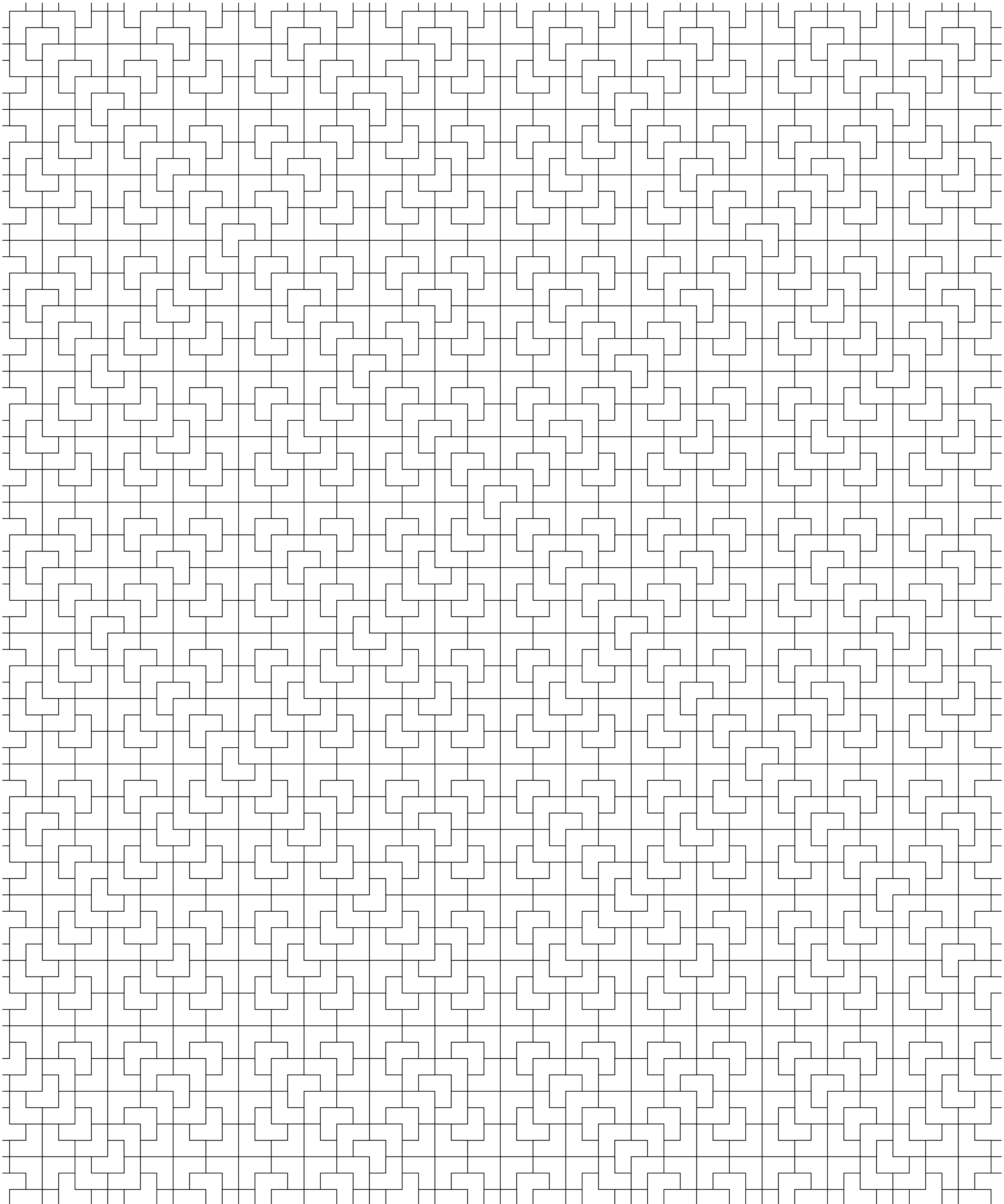


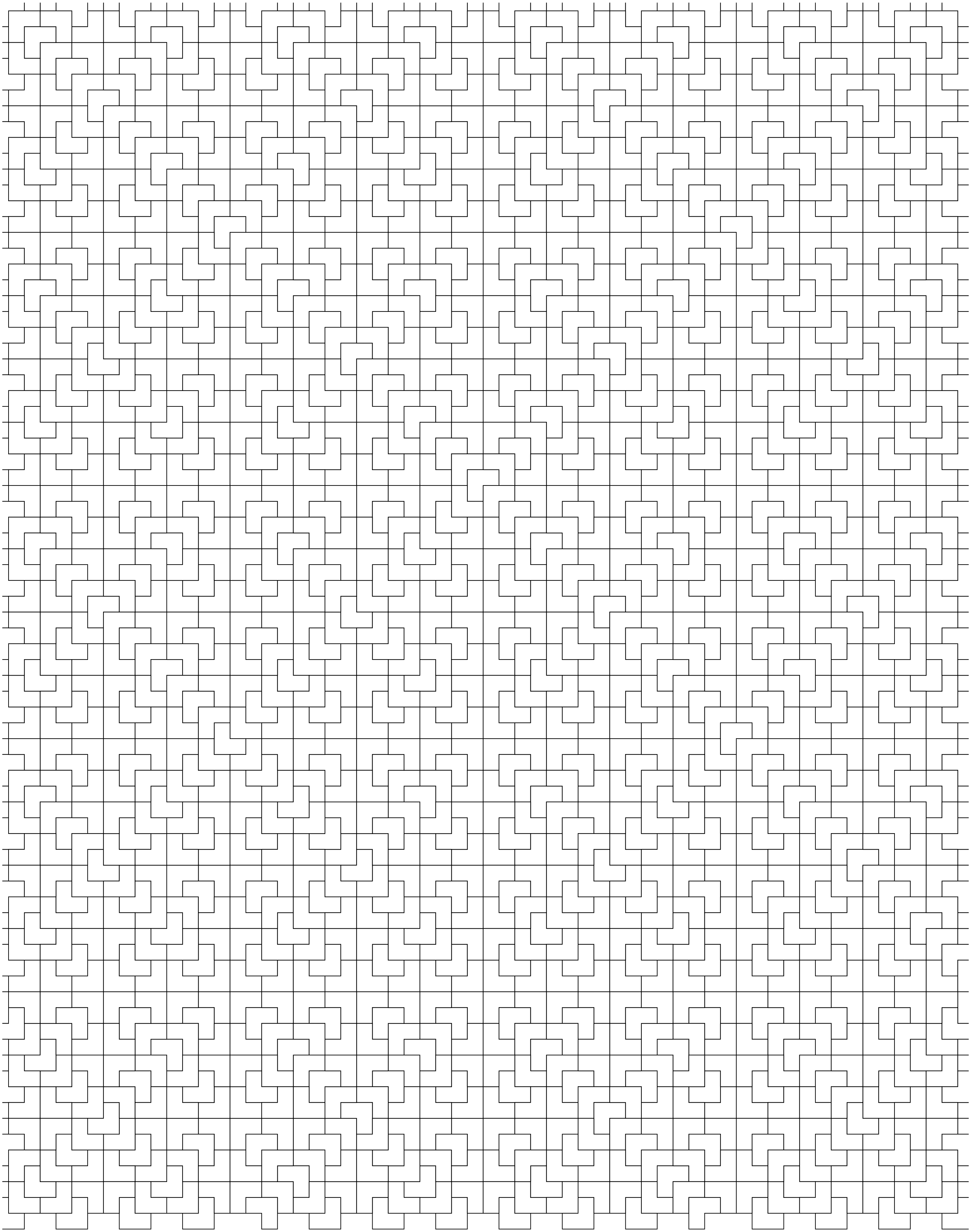
etc

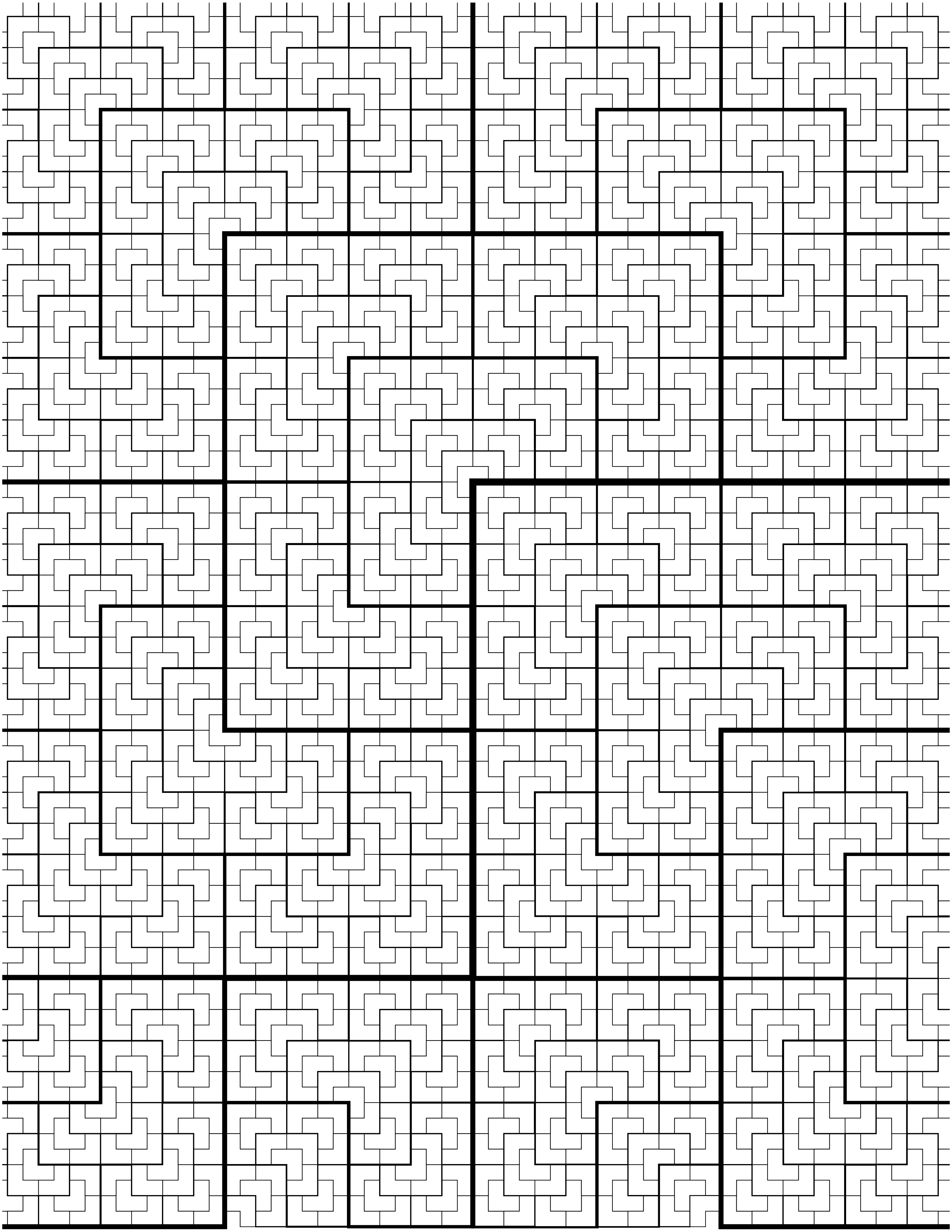
Substitution tilings

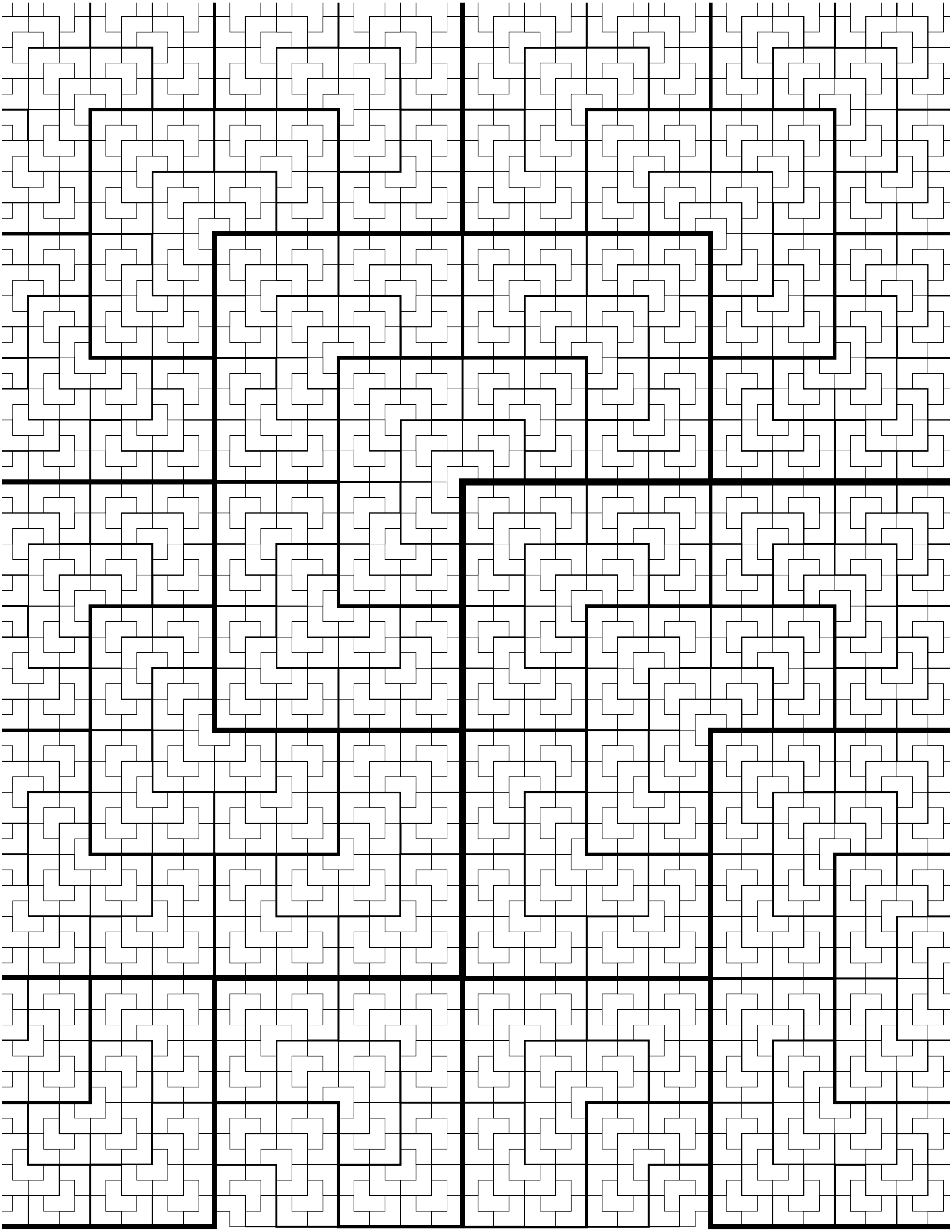
a great way of building non-periodic structure:











Substitution tilings are generally:

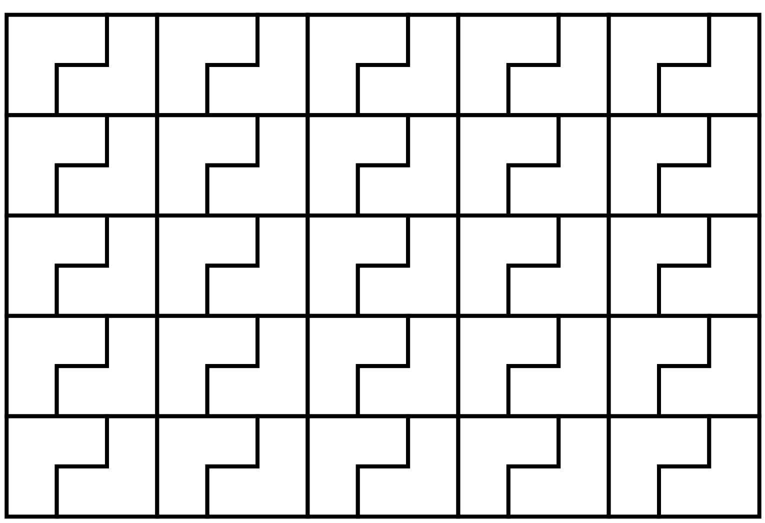
hierarchical

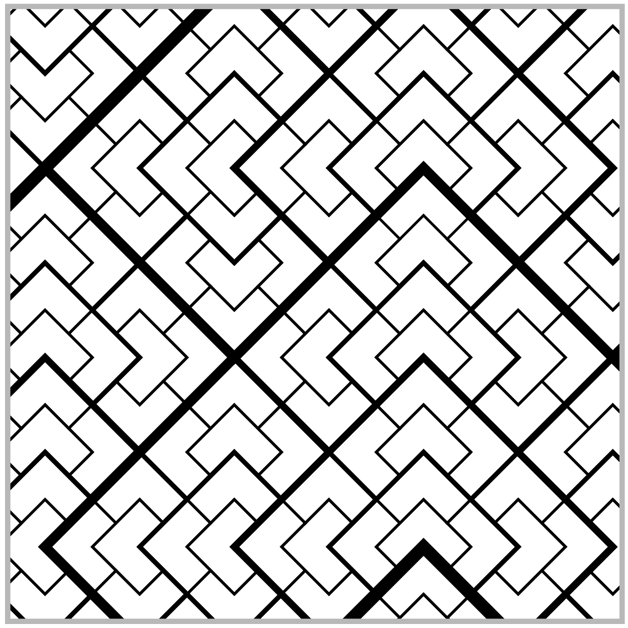
non-periodic

repetitive

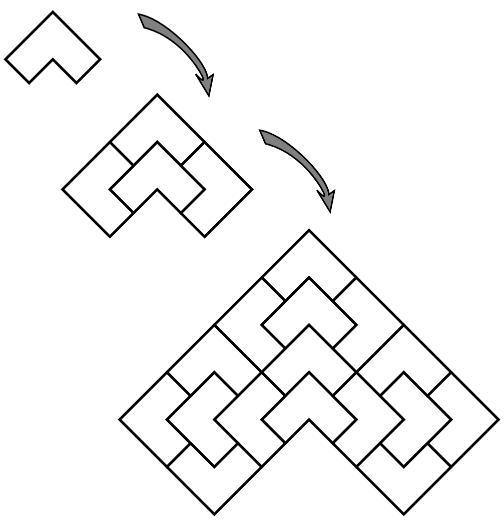
but the tiles themselves are *not* aperiodic

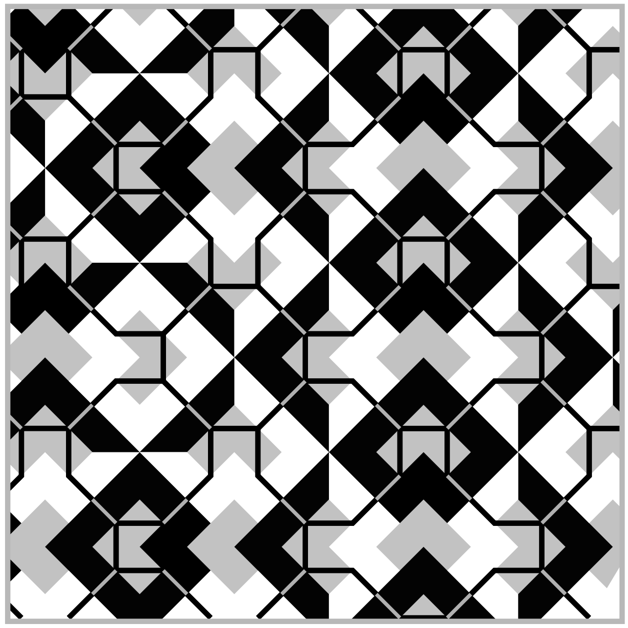
For example, the L-tiles *can* tile periodically.



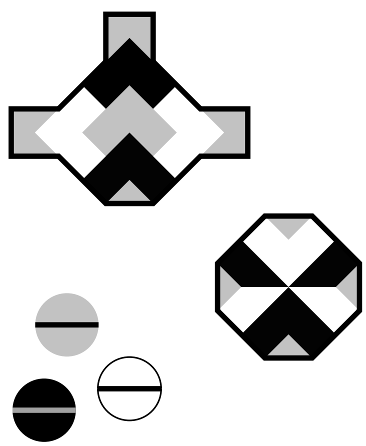


A substitution tiling





A matching rule tiling



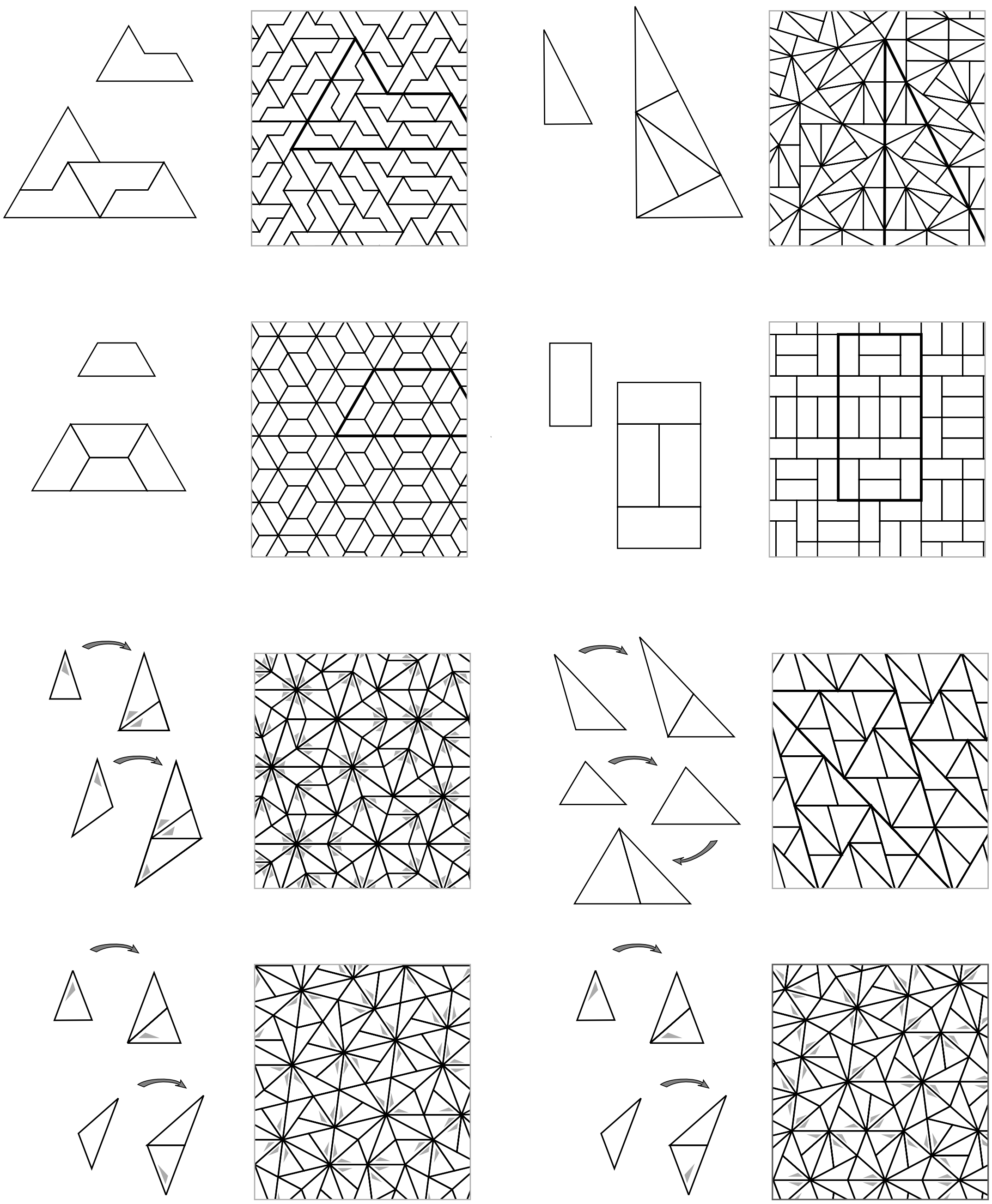
Since the tiles with matching rules *must* precisely reproduce the structure of the substitution tiling, we say these tiles *enforce* the substitution.

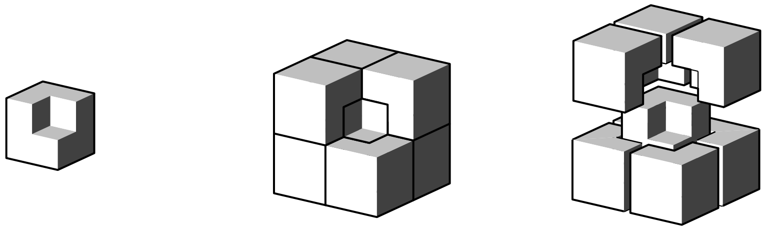
And these tiles must be aperiodic.

big question:

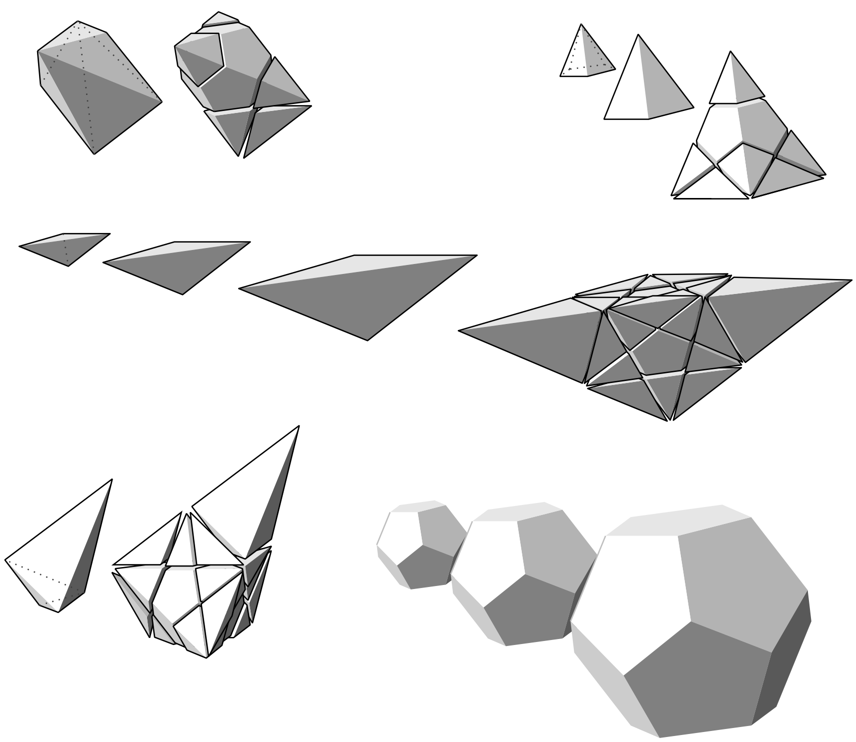
What substitution tilings can be enforced by matching rules?

There are lots of crazy examples
of substitution tilings:

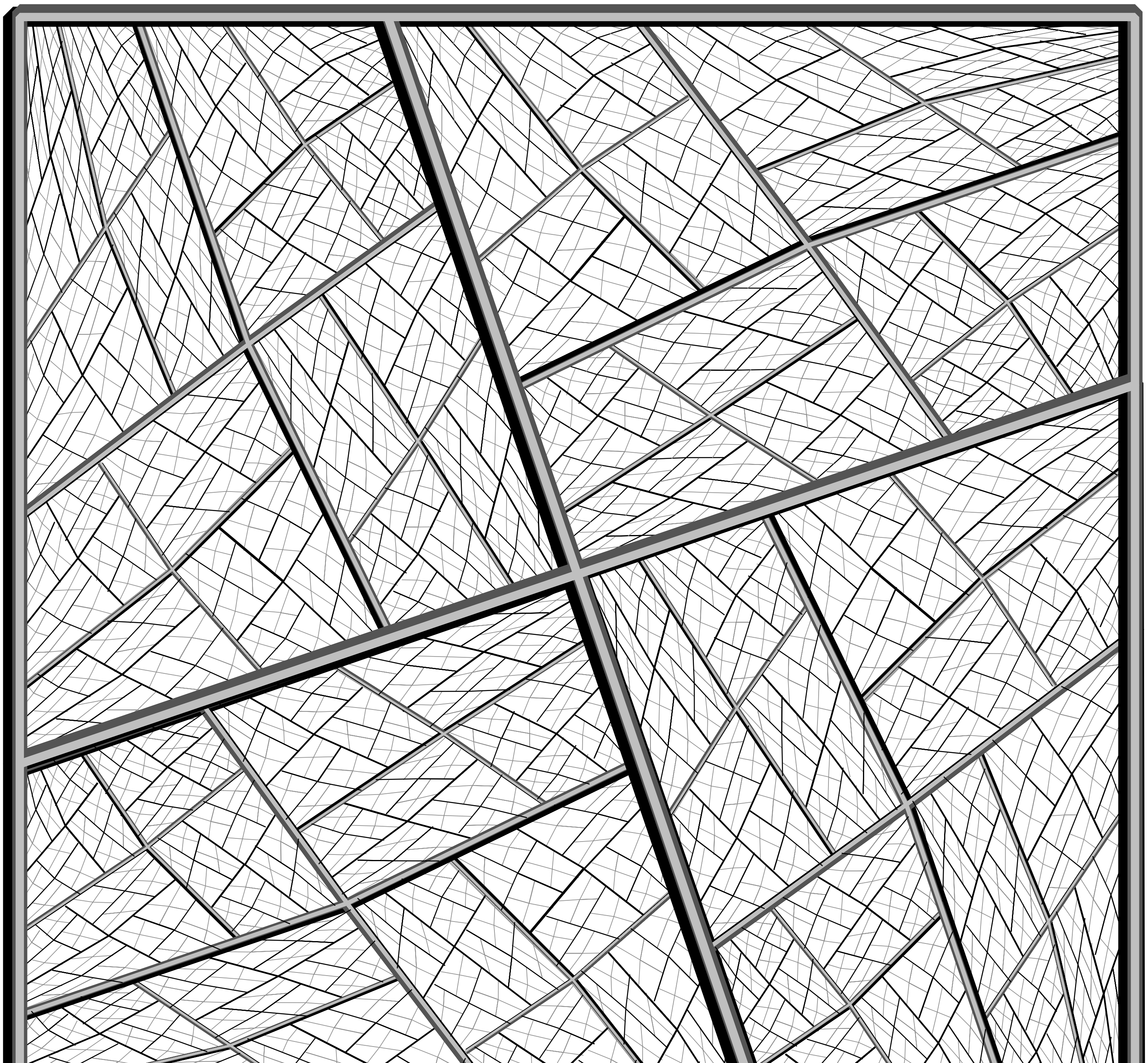




There are examples in all higher dimensions



This example gives rise to "dodecafoam"



In fact, no one yet knows just what substitution tilings are possible. No complete classification has been found, even in the plane.

But our question remains:

What substitution tilings can be enforced by matching rules to make an aperiodic set of tiles?

only a few examples were known . . .

Theorem: (G-S) *Every substitution tiling, in any dimension, (*) can be enforced by matching rules.*

As a corollary, this theorem produces infinitely many different aperiodic tilings.

(* up to a very mild technical condition satisfied by all known substitution tilings)

Issues in the proof:

the matching rule tilings must be *self-organizing*

information must be *locally finite*

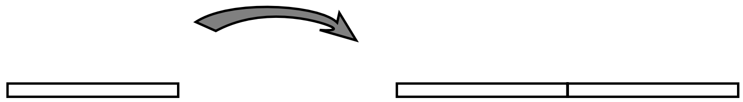
(how is the full hierarchy to be stored/encoded?)

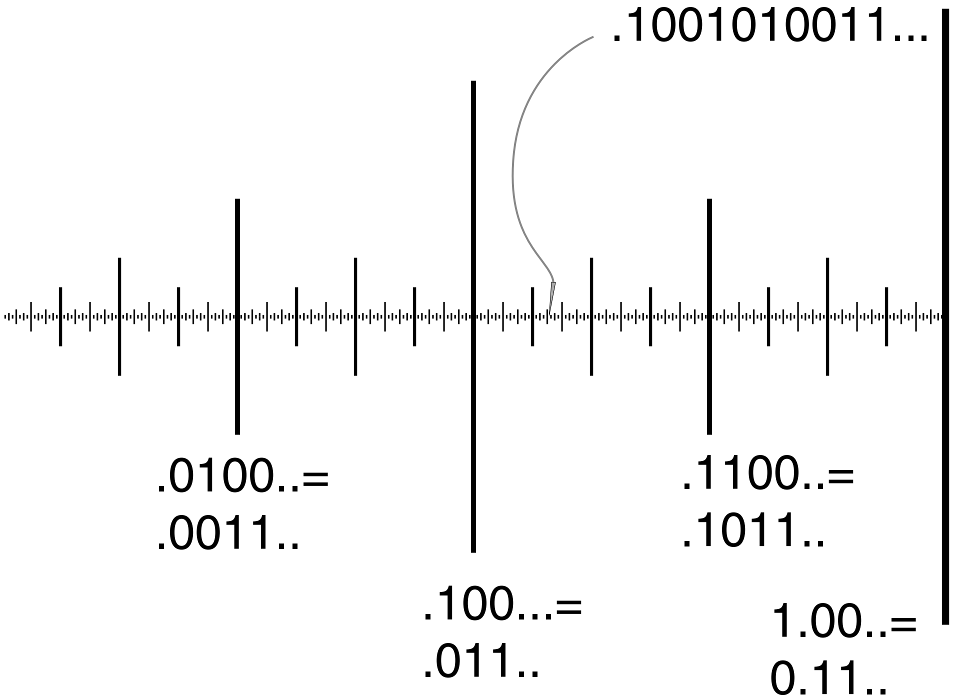
information must be *transmitted arbitrarily far*

(over self-organizing transmission lines!)

& just what *is* the desired structure anyway??

A FAMILIAR EXAMPLE OF A SUBSTITUTION TILING



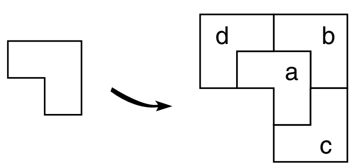


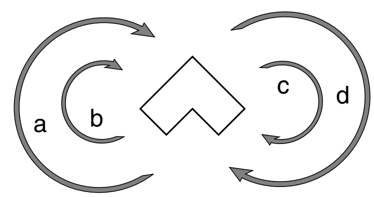
addresses can locate:

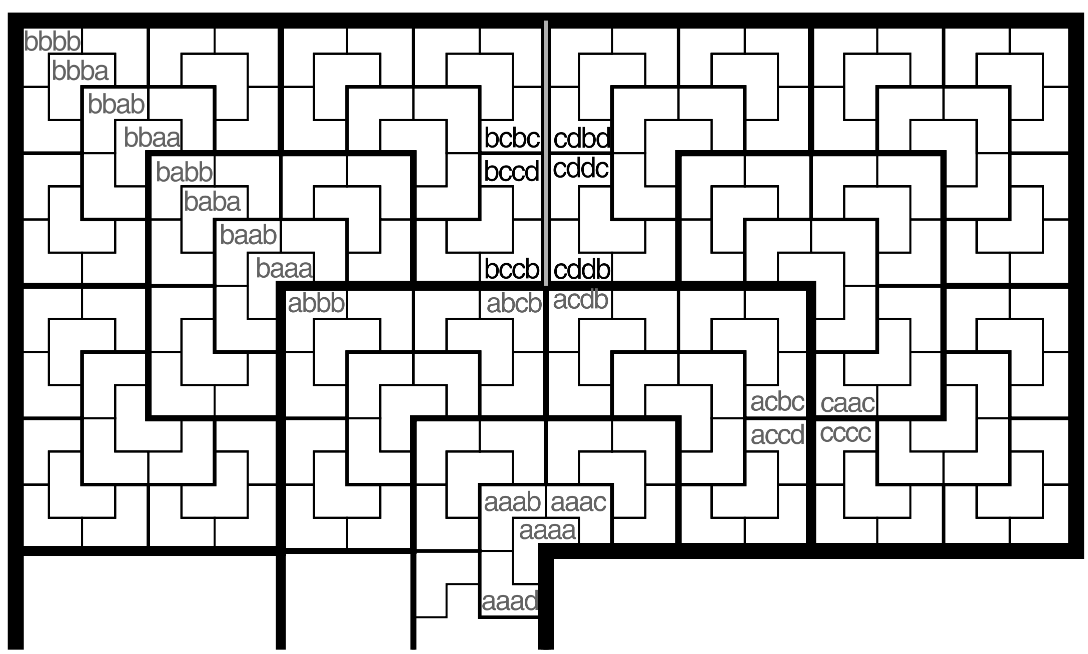
points within tiles

tiles within supertiles

points in entire tilings







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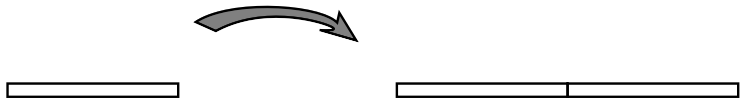
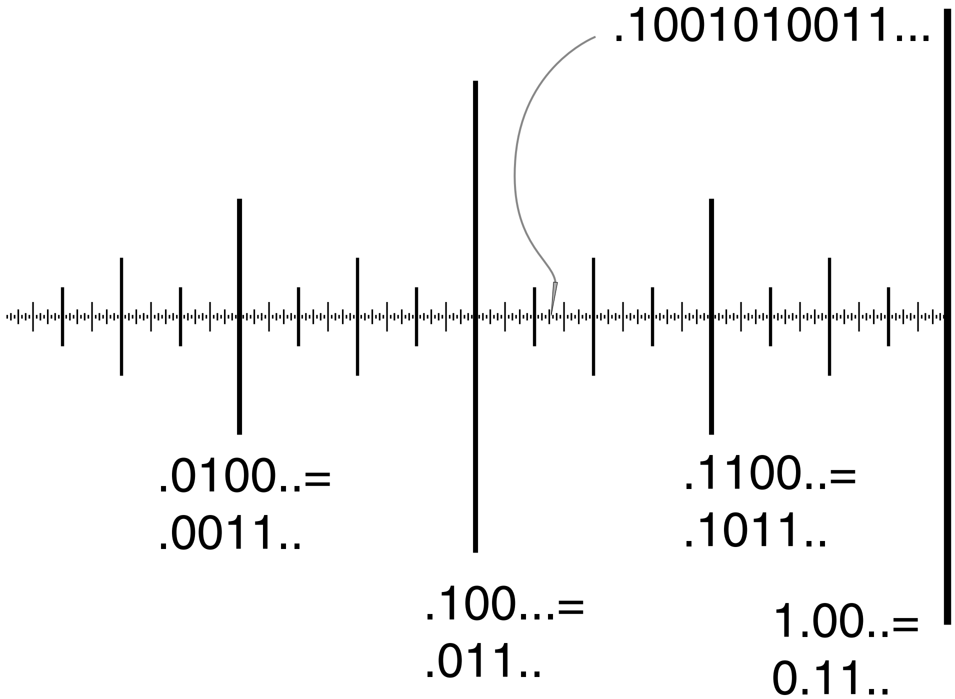


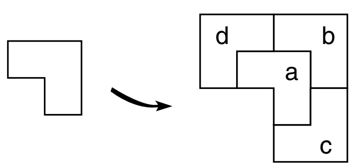


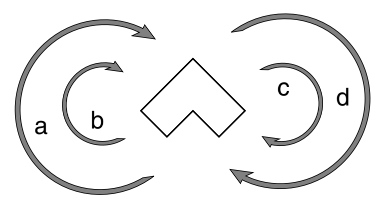
addresses can locate:

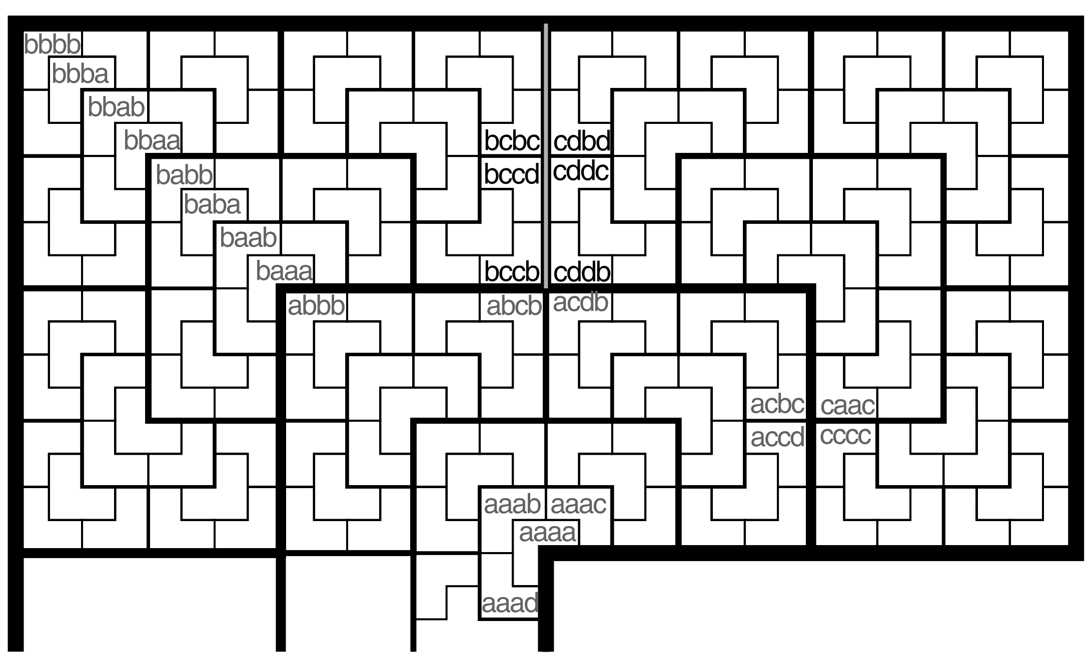
points within tiles

tiles within supertiles

points in entire tilings







from this we can recover the hierarchy precisely