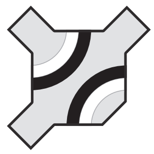
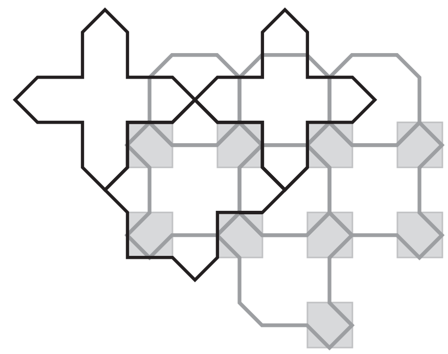


The trilobite and crab are an aperiodic pair of tiles.

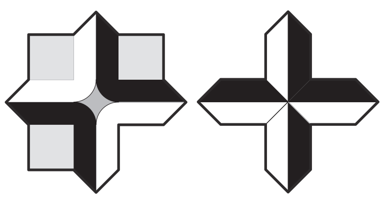
A full explanation.

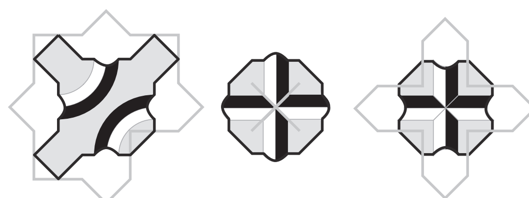
Chaim Goodman-Strauss, October 2015

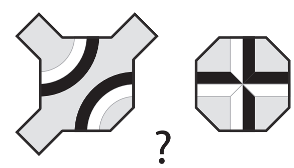


This pair of tiles is among the simplest aperiodic sets known. A proof they are aperiodic can be found and checked by hand, but for such a small set, will seem absurdly detailed, as in these notes. This proof complexity contrasts with two simpler variations *A small aperiodic set of planar tiles, Eur. J. Combin. (1999)*:

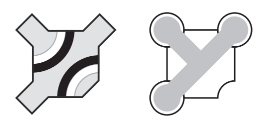
At left below, the ``trilobite and cross”, just two tiles quickly proven aperiodic, but meeting with tip-to-tip matching rules. As suggested at right above, we may transform this pair into three tiles, with more usually accepted edge-to-edge matching rules. The three tiles are drawn at middle below —their areas may be taken as 1, ε^2, ε.



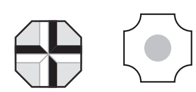




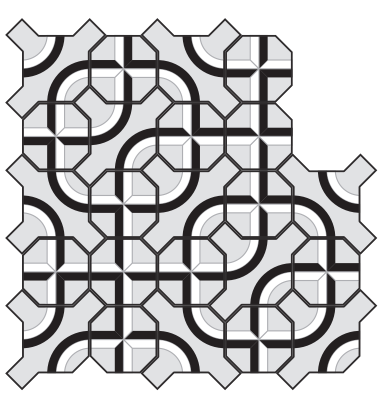
The proof that the first two sets are aperiodic and MLD, is fairly simple (see APT). In the second set, two of the three tiles are nearly identical. Yet conflating them, producing the pair at right, considerably complicates the proof of their aperiodicity.


trilobite

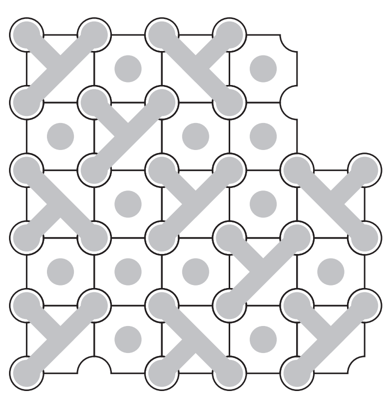
At right, we abbreviate our notation. The centers of all tiles must lie on a lattice; it suffices to specify the location and orientation of the trilobite tiles.

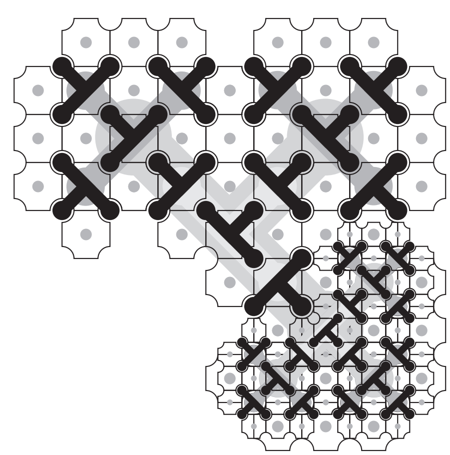

cross

Thus we draw



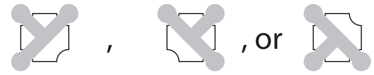
as

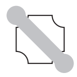
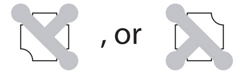




Our proof is essentially that this structure must be enforced by the matching rules.

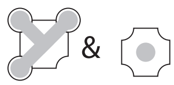
We further abbreviate:

 to mean 

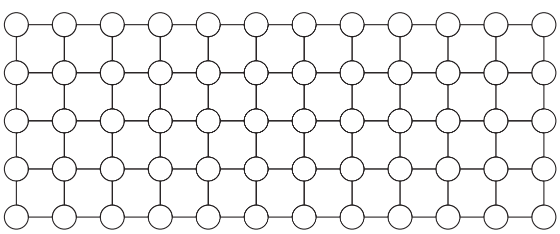
 to mean 

and  to mean any trilobite

In the following, solid **black** means given, implying gray, and numbers are in order of logical implication. **Green** means reduction to an earlier case. **Blue** means a choice. **Red** means a contradiction.

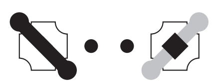
Our Axioms: with 

(A) we must fill every cell and every corner of

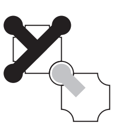
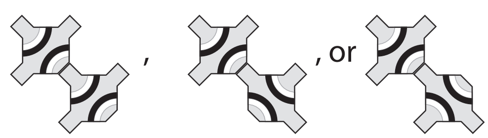


(some scratch paper like this is in the back of the packet)

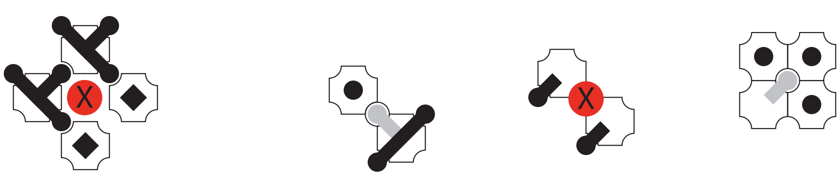
(B) By parity, if two trilobites are separated only by crosses, they must be perpendicular.

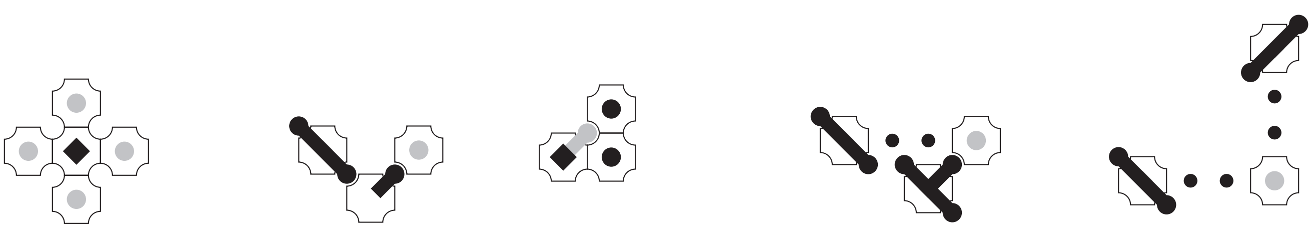


We have a number of minor lemmas, such as the following.
(You may find scratch paper provided on the last page to be helpful.)

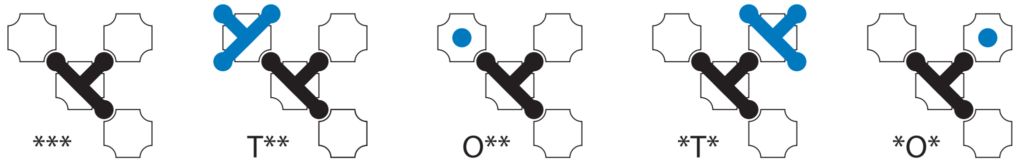
 meaning that adjacent to  we must have 

Similarly

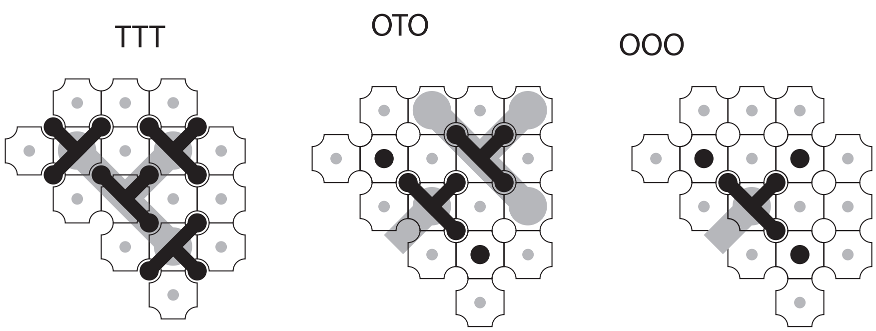




We consider the possible tiles surrounding a trilobite:

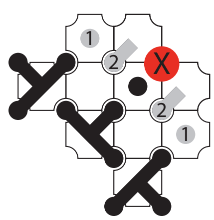


TTT, OTO and OOO arise within the heirarchical structure we seek.

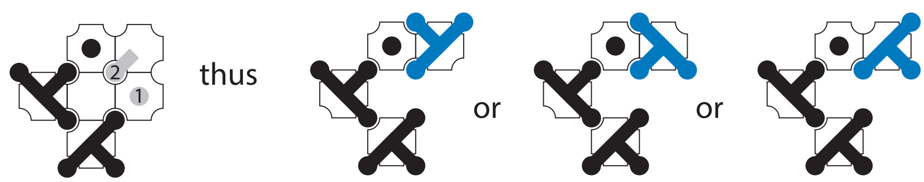


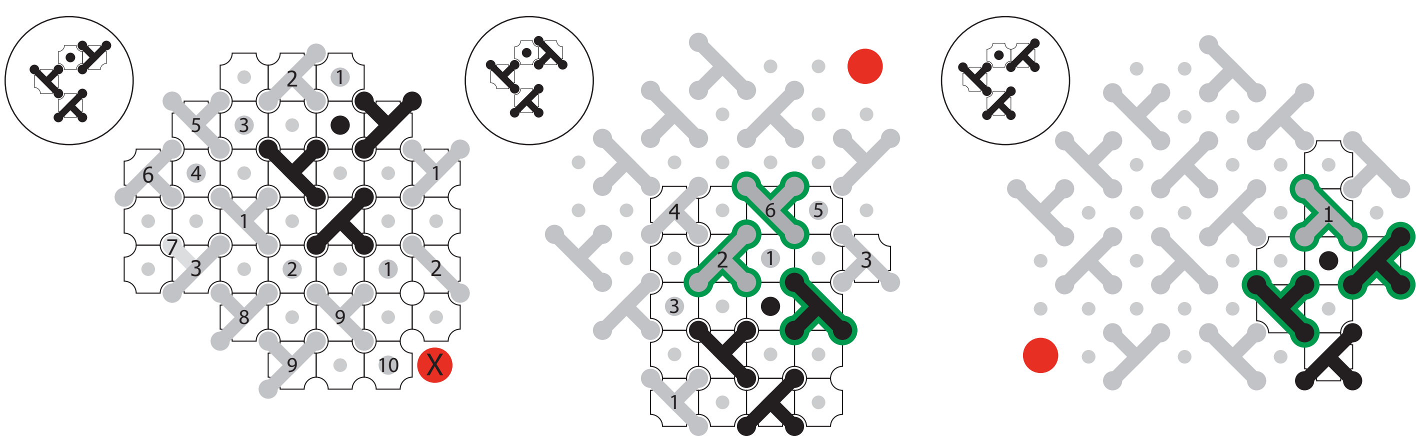
but OTT, TTO require special care and we seek to entirely forbid TOT and OOT, TOO.

TOT Is forbidden

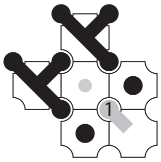


*OT is forbidden

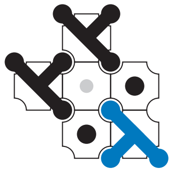




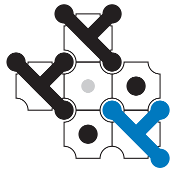
Chains of *TO's are forbidden



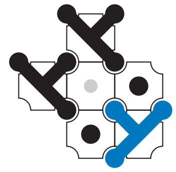
thus

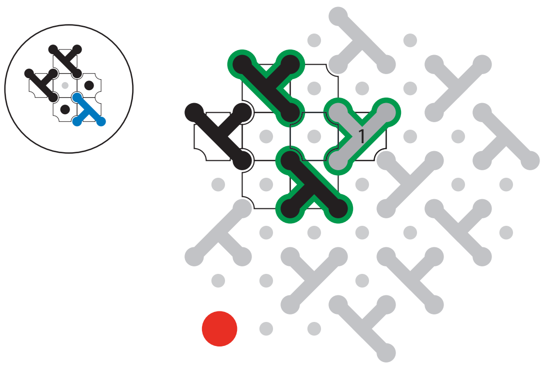


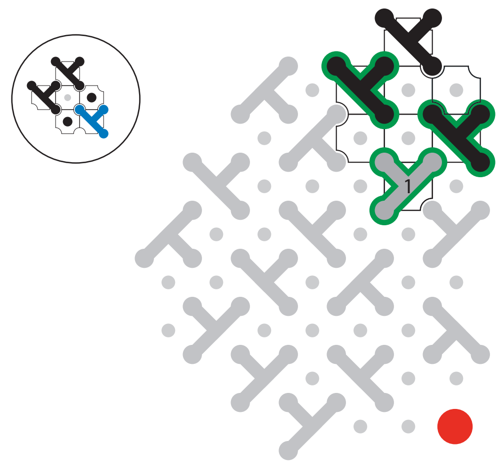
or

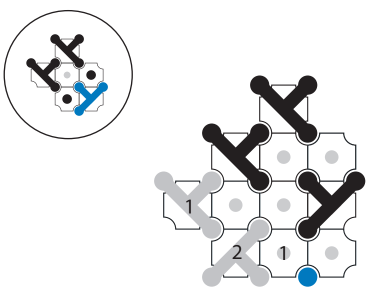


or

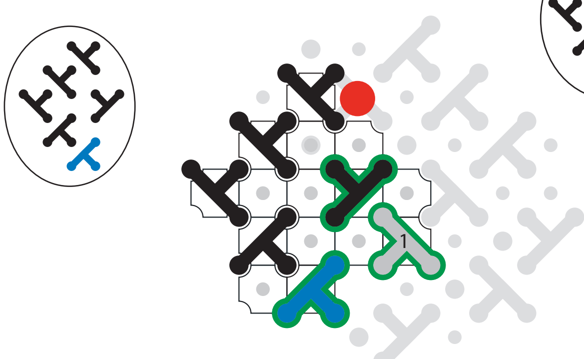


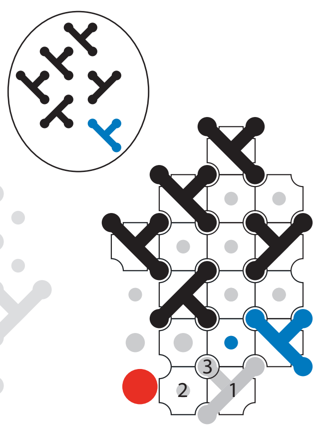


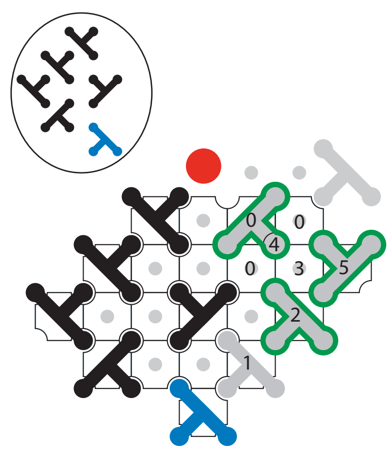




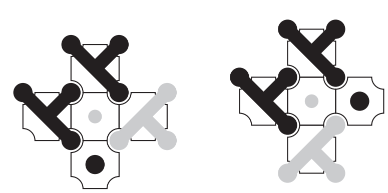
has further subcases:



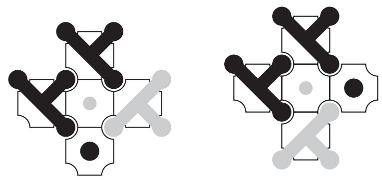




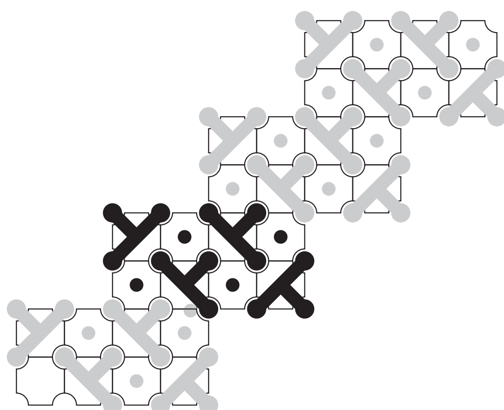
Therefore:



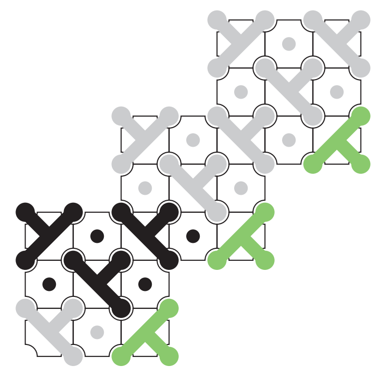
Therefore:



Therefore any OTT or TTO, should they arise, can only occur in an unending chain of OTT and TTO's.
Any tiling with such a chain, exactly corresponds to a a tiling with an unending chain of TTT and OTO's.



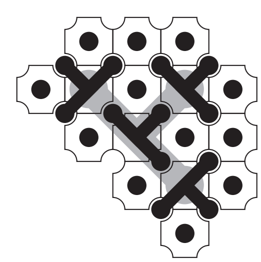
because we may slide half of our tiling along this infinite chain.



We may thus consider tilings in which each trilobite is TTT, OTO or OOO, for if any such tiling is non-periodic, so too will be any tiling that may include an infinite chain of OTT and TTO's.

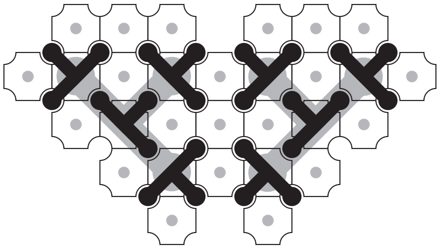
Suppose then that each trilobite is one of TTT, OTO or OOO. Then each trilobite, either is TTT (and thus is not one of the T's of one of its neighbors), or is one of the T's for one if its neighbors.

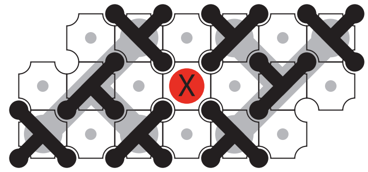
Each trilobite lies in an unique TTT and ``deflation" is possible.

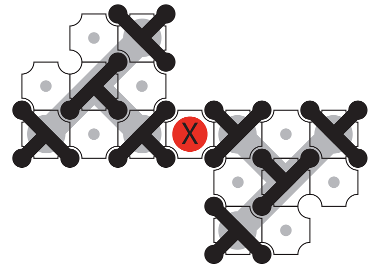


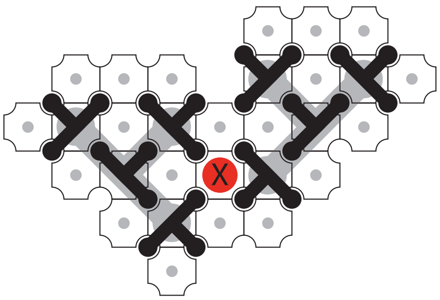
However, we must check that these deflated tiles still have the same combinatorial structure.
In particular, can supertrilobites meet in new ways?

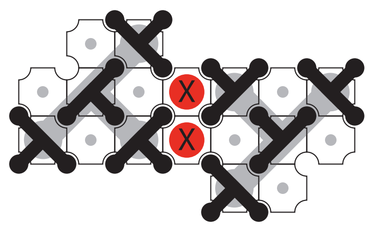
We have several cases to check. The first satisfies the axioms, as we would like, and several more do not:

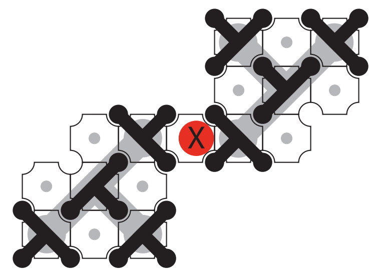


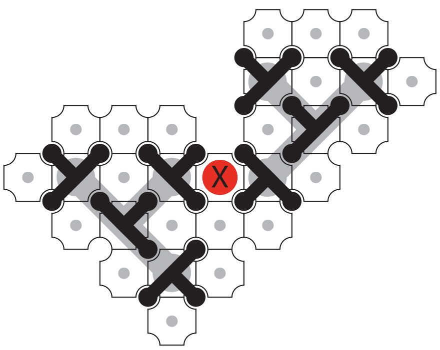


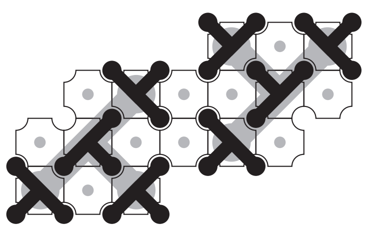




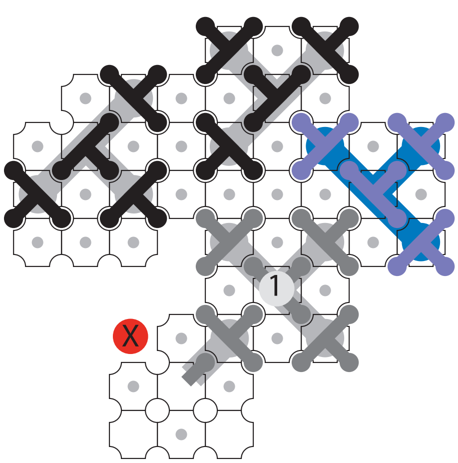


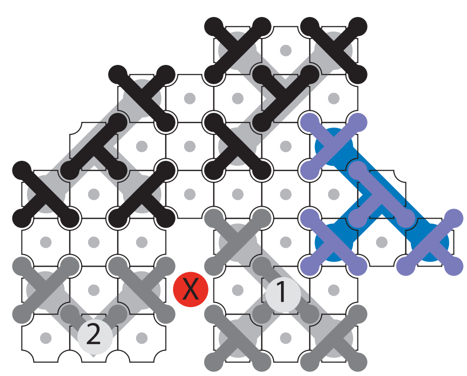


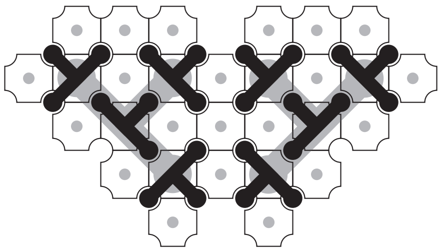




This remaining case
has further subcases.





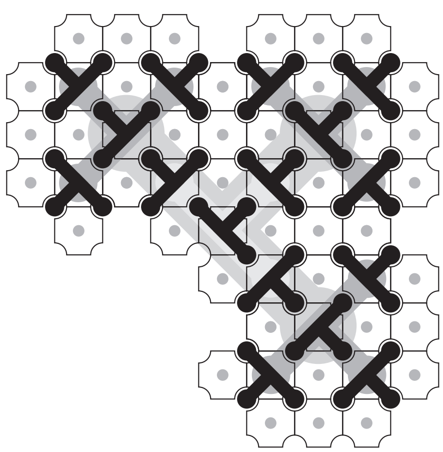


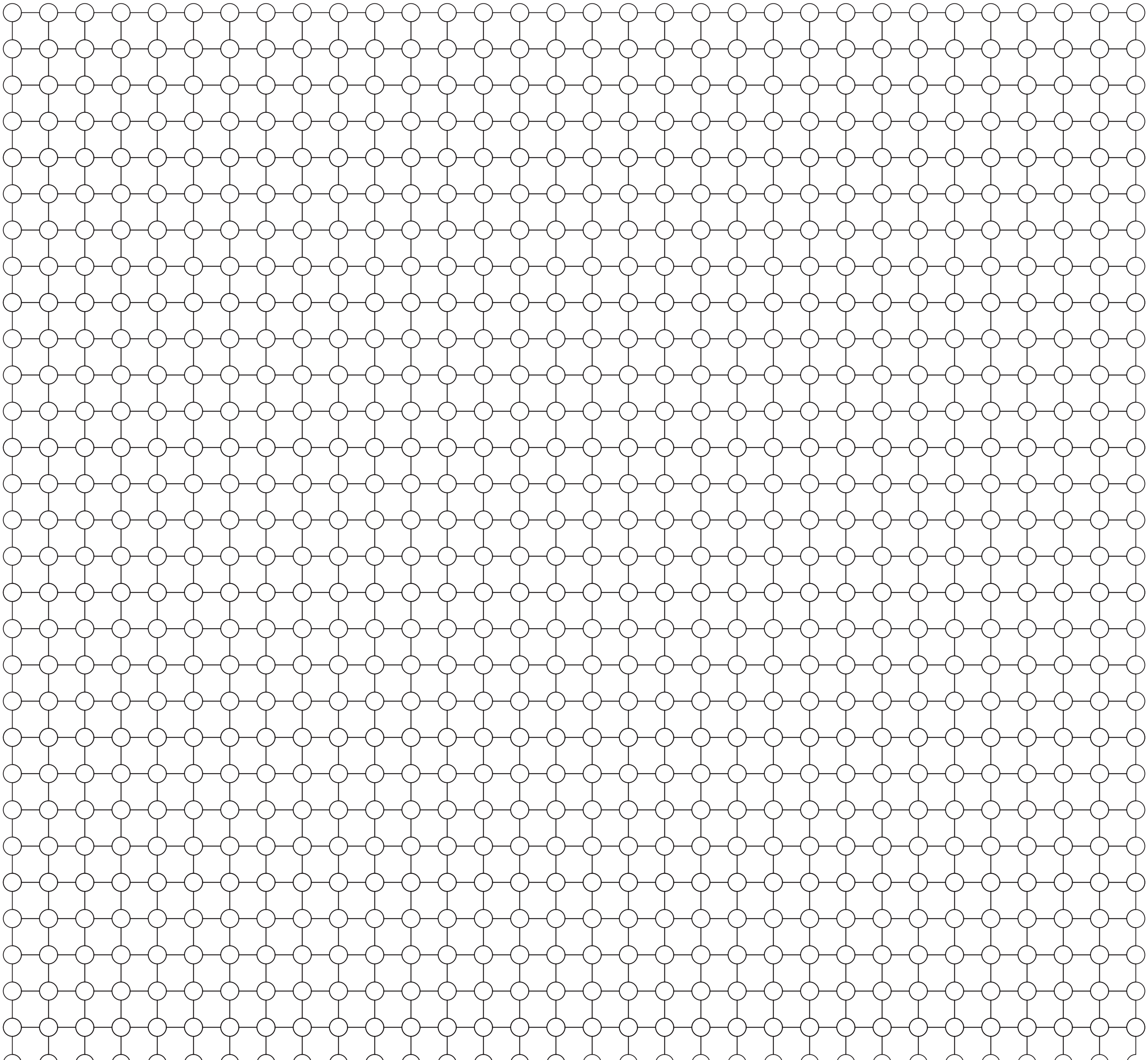
Consequently, the induction proceeds:

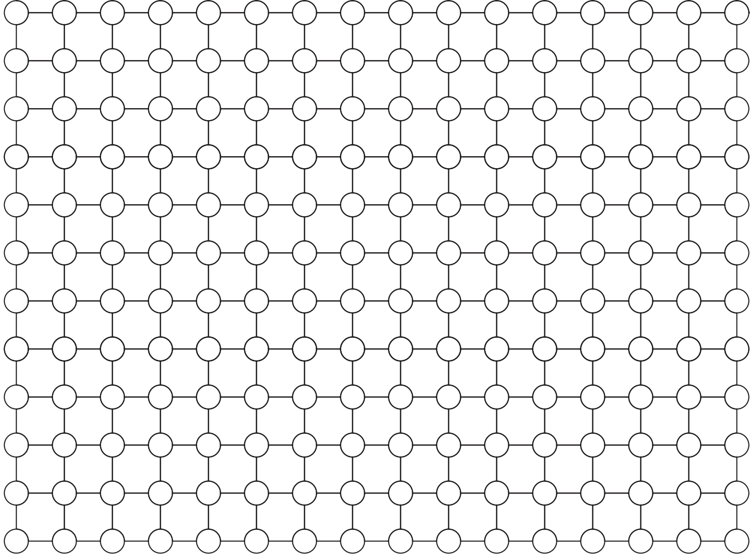
Every tiling by the trilobite and cross corresponds to a tiling by a larger trilobite and cross, perhaps after applying a slide down an infinite diagonal.

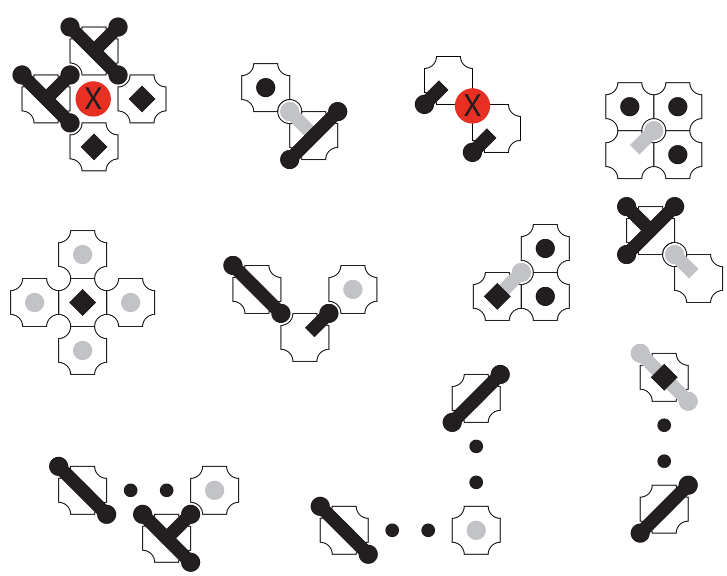
The argument is as in

A small aperiodic set of planar tiles
C Goodman-Strauss
European J. Combinatorics,
1999





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