# Levels of Infinity

Are there as many even numbers as there are counting numbers?

Galileo (yes that one) said this is a paradox: on the one hand, there are obviously *half* as many even numbers as counting numbers.

1 2 3 4 5 6 7 8 9 10 11 12 13 ...

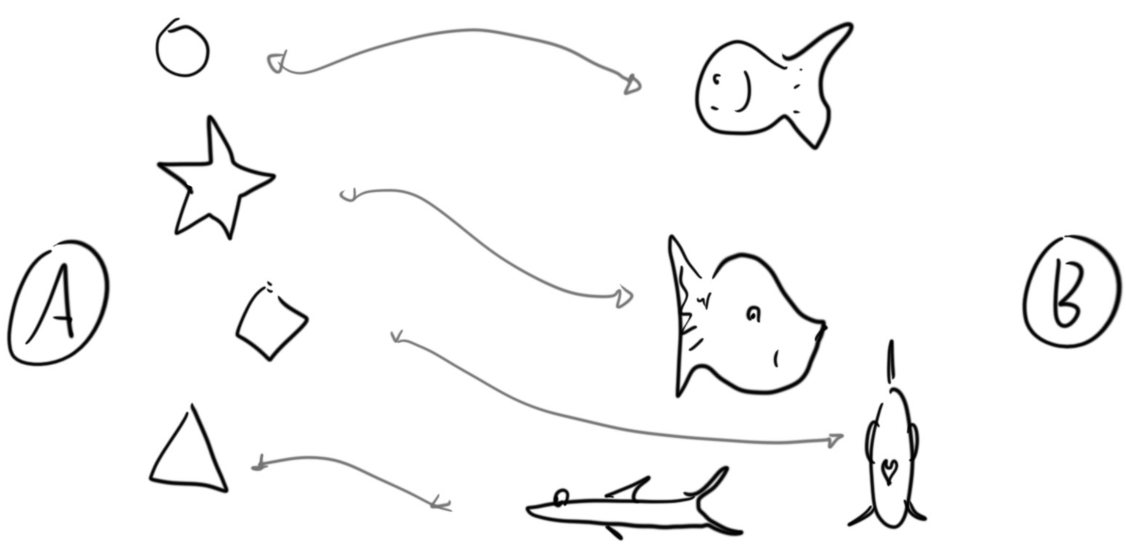
On the other hand, maybe there are *exactly as many*, because we can match each counting number up with an even number:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| counting numbers | 1 | 2 | 3 | 4 | 5 | 6 | 7 | ... |
| | $\updownarrow$ | $\updownarrow$ | $\updownarrow$ | $\updownarrow$ | $\updownarrow$ | $\updownarrow$ | $\updownarrow$ | ... |
| even numbers | 2 | 4 | 6 | 8 | 10 | 12 | 14 | ... |

To make sense of this, we just have to clarify, just

*What do we* mean *when we say that two infinite sets are "the same size"?*

Almost 300 years later, Georg Cantor pointed out that finite sets are the same size exactly when their contents can be matched up, one for one. For example, there are the same number of shapes in Set A as there are fish in Set B, because each fish is matched to a shape and vice versa.




In fact, when we say there are "four" fish, or "four" shapes, we mean that these sets can be matched up with a special set of number words, {*one, two, three, four*}.

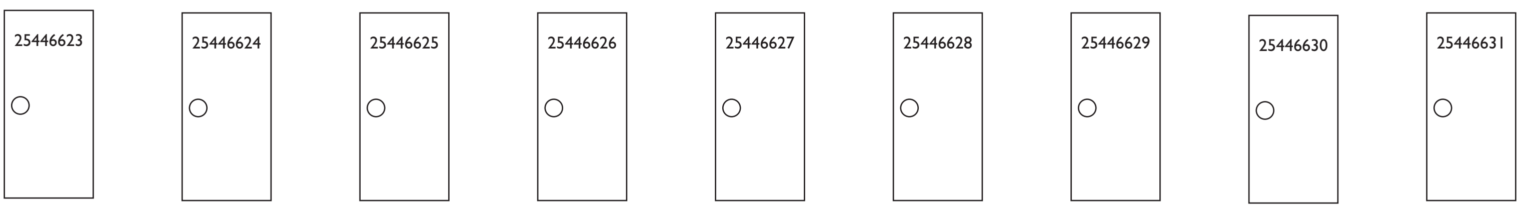
Cantor suggested *Why not think of this the same way as for infinite sets?* We simply say two sets, whether finite or infinite have the same cardinality ("size") if all their elements can be matched up, one-for-one.

So in the drawing, the set of shapes has the same cardinality as the set of fishes. In the same way, we answer Galileo's paradox, and sensibly declare that set of even numbers has the same cardinality as the set of counting numbers.

Sounds good? Well, here come the surprises!!

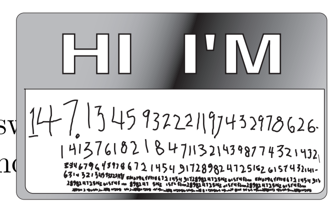
## The Hilbert Hotel

Imagine a hotel[1] with rooms numbered 1, 2, 3, 4, 5, . . . , on and on forever, and suppose each room is occupied.




• Suppose another guest shows up and would like to check in. How can the hotel make room by moving the guests already there? *(The hotel has to actually give room numbers to all of the guests! It's not legit to just ask the guest to go to the end and find the last room— what last room?)*

• What if five new guests appear and want to check in?




• How about a billion?

• What if a counting number's worth of new guests ask for rooms? To answ question, you must give a rule for where each guest already in the hotel mo and for where each new guest moves into.

• Can the hotel accomodate as many new guests as there are real numbers?

A set is *countably infinite*[2] if it has the same cardinality as the set[3] of counting numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$— in other words, if its elements can be matched one-to-one with the counting numbers. This term makes sense: a set is countably infinite if its elements can be listed in an infinite list.

[1] Named for the great mathematician David Hilbert, an early champion of Cantor's idea.

[2] A countable set, officially, is any finite or countably infinite set. Really, when most people (me included) say "countable", we mean "same cardinality as $\mathbb{N}$", countably infinite.

[3] Unfortunately, there is not a standard meaning for $\mathbb{N}$. Some mathematicians define $\mathbb{N} = \{0, 1, 2, ...\}$ and others define $\mathbb{N} = \{1, 2, 3, ...\}$. Oh well. Just have to say which you mean every time. But the *cardinality* of these two sets is the same. Why?

Show that these sets are countable:

• $\{5, 6, 7, 8, \dots\}$

• The set of even numbers.

• The set of prime numbers.

• The set of square numbers.

• Now for an interesting example. The set of all integers $\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, \dots\}$ is countable! What is a way to match these up with the counting numbers?

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| counting numbers | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | … |
| | ↕ | ↕ | ↕ | ↕ | ↕ | ↕ | ↕ | ↕ | ↕ | ↕ | ↕ | ↕ | |
| the integers | 0 | __ | __ | __ | __ | __ | __ | __ | __ | __ | __ | __ | … |

• Here is the first real surprise: The set of all *rational* numbers is countable! This is amazing, because the rational numbers are "dense" on the real number line— in any interval, no matter how small, there are infinitely many rational numbers! And yet they can be listed out, and matched with the counting numbers. To simplify matters, we'll just show the positive rational numbers are countable, using Cantor's original method. We list the positive rationals in this order, striking out any duplicates

| sum of numerator and denominator | listed in order of numerator | | | | | |
|---|---|---|---|---|---|---|
| 2 | 1/1 | | | | | |
| 3 | 1/2 | 2/1 | | | | |
| 4 | 1/3 | ~~2/2~~ | 3/1 | | | |
| 5 | 1/4 | 2/3 | 3/2 | 4/1 | | |
| 6 | 1/5 | ~~2/4~~ | ~~3/3~~ | ~~4/2~~ | 5/1 | |
| 7 | 1/6 | 2/5 | 3/4 | 4/3 | 5/2 | 6/1 |
| ⋮ | | | ⋱ | | | |

This gives a matching with the counting numbers:

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| counting numbers | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | … |
| | ↕ | ↕ | ↕ | ↕ | ↕ | ↕ | ↕ | ↕ | ↕ | ↕ | ↕ | ↕ | |
| rational numbers | 1/1 | 1/2 | 2/1 | 1/3 | 3/1 | 1/4 | 2/3 | 3/2 | 4/1 | 1/4 | 5/1 | 1/6 | … |

Complete a few more rows of the table and extend the list!

## The Reals Are Not Countable!

So you might be thinking *every* infinite set is countable. But Cantor showed that's just not so:

Cantor showed that there are vastly more *real* numbers than there are counting numbers. You simply cannot list out the reals!!

Any infinite list of real numbers must be missing something; let's see how with an example. For simplicity, suppose we have a list of real numbers between 0 and 1, like this one:

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0. 8 | 4 | 6 | 9 | 6 | 3 | 8 | 3 | 6 | ... |
| 2 | 0. 4 | 0 | 2 | 3 | 2 | 6 | 5 | 4 | 8 | ... |
| 3 | 0. 4 | 7 | 5 | 7 | 4 | 5 | 1 | 4 | 0 | ... |
| 4 | 0. 6 | 5 | 4 | 0 | 8 | 0 | 7 | 8 | 3 | ... |
| 5 | 0. 7 | 1 | 8 | 4 | 8 | 6 | 6 | 3 | 4 | ... |
| 6 | 0. 1 | 8 | 5 | 4 | 1 | 5 | 3 | 5 | 4 | ... |
| 7 | 0. 0 | 7 | 3 | 9 | 1 | 0 | 5 | 3 | 8 | ... |
| 8 | 0. 2 | 9 | 2 | 7 | 3 | 1 | 0 | 1 | 4 | ... |
| 9 | 0. 9 | 2 | 7 | 1 | 9 | 6 | 1 | 5 | 8 | ... |
| ⋮ | | | ⋱ | | | | | | | |

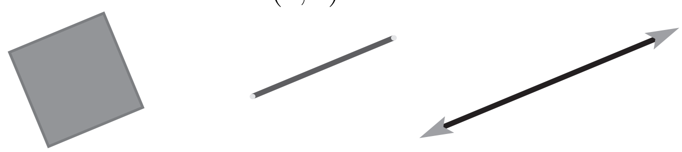
But then the real number 0.916196629.... is not on the list — it differs from the first number on the list in the first digit, the second number in the second digit, and so on.

No matter what the original list was, we could *always* cook up a new real number that's not on the list. NO list could ever have been a complete listing of all of the reals!!

In fact, there are *vastly* many more reals than there are counting numbers — the counting numbers are a *neglible*, "measure 0" subset of the reals.

## Other surprising examples

The cardinality of the set of reals is the same as that of any interval of reals! For example, there are just as many numbers in the interval $(-1, 1)$ as there are reals! The function $f(x) = x/(x^2 - 1)$ matches the numbers $(0, 1)$ one-for-one with all of the reals.[4]



The cardinality of the entire plane is the same as that of the line! More simply, we can match the points in the unit square with the points in the unit interval. For example, we match

(0.1234..., 0..9876...) ↔ 0.19283746...

## An Infinity of Infinities!

For any set $A$, the power set $\mathcal{P}(A)$ is the set of all subsets of $A$. For example, if $A = \{a, b, c\}$, then $\mathcal{P}(A)$ consists of the eight $(2^3)$ subsets, $\{\}, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}$ and $\{a, b, c\}$. There are $2 \times 2 \times 2$ subsets because — each of the three elements of $A$ can be in a subset or not. In this way, for any finite set $A$ of cardinality $n$, the cardinality of $\mathcal{P}(A)$ is $2^n \neq n$.

This is true in general. Cantor proved that the cardinality of *any* set $A$ is strictly less (in a way we'll describe in a moment) than the cardinality of its power set $\mathcal{P}(A)$. So starting with $\mathbb{N}$, we have an *infinity* of infinite cardinalities (!!)

$$\mathbb{N}, \quad \mathcal{P}(\mathbb{N}), \quad \mathcal{P}(\mathcal{P}(\mathbb{N})), \quad \mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{N}))), \quad \mathcal{P}(\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{N})))), \ldots$$

Proof: Let $A$ be a set, and suppose there were a one-to-one matching with the elements of $\mathcal{P}(A)$ with all of the subsets of $A$. We will show this leads to a contradiction, and so no such matching could exist.

So suppose each element $s$ of $A$ is matched with some subset $f(s) \subset A$. Any $s$ might be an element of $f(s)$, or not, depending on the matching. We define the subset

$$C := \{\text{the elements } s \text{ of } A \text{ so that } s \text{ is } \textit{not} \text{ an element of } f(s)\}$$

We've supposed (for contradiction) that every subset of $A$ has been matched up — for all $S \subset A$, there is some element $s$ of $A$ with $f(s) = S$. In particular, there is some $c$ so that $f(c) = C$.

[4]For any real number $y$, there is a unique $x$ in $(0, 1)$ with $f(x) = y$, namely $x = (1 - \sqrt{1 + 4y^2})/(2y)$.

BUT, $c$ cannot be in $C$: If $c$ were in $C$ then $c$ is not in $f(c) = C$, a contradiction.

On the other hand if $c$ is not in $C$ then $c$ is not in $f(c)$, and so $c$ *is* in $C$, again a contradiction.

So $c$ cannot be in $C$, and cannot not be in $C$— the issue is that this $c$ could not have existed in the first place.

The cardinalities of $A$ and $\mathcal{P}(A)$ could not have been the same!

## Comparing infinities

More generally, we can define the relationship that "set $A$ has at least the cardinality of set $B$" iff there is a one-to-one matching from the elements of $B$ to at least some of the elements of $A$.

It's not obvious that the cardinality of any two sets could be compared in this way, that all of the possible cardinalities can be put in order. In fact, that this is so is an axiom, equivalent to the "Axiom of Choice", "Zorn's Lemma" and "The Well-Ordering Principle".

The Cantor-Schroeder-Bernstein Theorem shows that if $A$ has (by this definition) at least the same cardinality of $B$ and $B$ has at least the cardinality of $A$, then $A$ and $B$ have the same cardinality as each other. It is therefore meaningful to define "$A$ has *strictly* greater cardinality than $B$" if $A$ has a least the cardinality of $B$ but $B$ does not have at least the cardinality of $A$.

The proof of this theorem is subtle: given a one-to-one matching from the elements of $B$ to some of the elements of $A$, and another from the elements of $A$ to $B$, one must show there is a one-to-one matching between all of the elements of $A$ and all of the elements of $B$.

But this raises a big question:

It turns out that the set of reals has the same cardinality as $\mathcal{P}(\mathbb{N})$. But is there anything in between? It turns out, incredibly, that you can take this either way, introducing this as a new axiom, without introducing any contradiction — Paul Cohen won the Fields Medal for his proof the independence of "The Continuum Hypothesis" in 1966.

### Further reading

*Logicomix!* is up in the MoMath bookstore.

*Aha! Gotcha!* by Martin Gardner.