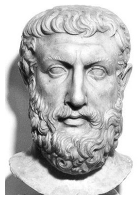
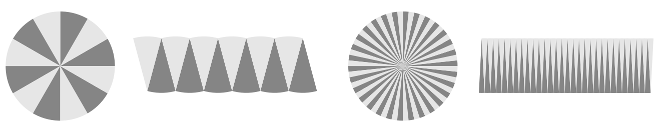
Short History of the Infinite



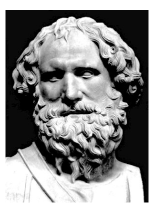
Zeno of Elea (~490BCE–430) was one of the first philosophers to confront the infinite, putting front and center issues that took millennia to resolve. For example, is half, plus half again, plus half again, ad infinitum, equal to a whole?

Despite misgivings, infinite arguments are just too useful to pass up! Ancient mathematicians computed formulas for the volumes and surface areas of spheres, cones and more complex shapes, paths of planets (as they understood them), the value of π, and much much more, all by using finer and finer approximations, finally limiting onto the true value. Contemporary calculus is no more than this idea taken to its full power.

Typical argument using an infinite process:

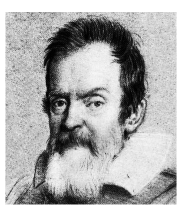


Theorem: The area of a circle is its radius times half its circumference
That is, if the radius is r, the circumference is $2\pi r$ and the radius is πr^2



Archimedes (287BCE-212), one of the greatest mathematicians of all time, used infinite arguments to great advantage, inventing a form of calculus 2000 years before Newton and Leibniz. Fully aware of Zeno's traps, his methods foreshadowed Cauchy's, millennia later.

Chinese mathematicians were similarly adept in arguments using the infinite. Using such methods, 祖冲之 (Zu Chongzhi, 429-500) was able to compute π accurately to seven decimal places, a feat not matched in the West for well over 1000 years.



By the 17th century, Western mathematicians had lost much of their concern with whether or not infinity made "sense" and just got to work. Galileo Galilei (1564-1642) used many infinite arguments in his astronomical computations.

At the age of just 24, Isaac Newton (1642-1727) finally grasped the true power of these kinds of methods, inventing the Calculus. His Principia is full of arguments using infinitely small quantities, summing them, taking ratios between them, etc, to dazzling, unprecedented effect. However he kept his methods secret....

.... until Calculus was reinvented independently by the great Gottfried Leibniz (1646-1727). A bitter rivalry ensued, splitting much of the mathematical world for another century!

Leibniz and Newton felt perfectly comfortable with such outrageous ideas as measuring the different sizes of infinitely small quantities—with great success! For example, the speed of an object is distance ÷ time. But what if speed is changing? What is the speed at a split moment? "Clearly" it is the infinitely small distance traveled in that moment (dy, say) ÷ the infinitely small time of that moment (dt, say) The expression dy/dt means just that: this ratio!

The 18th Century saw an explosion of mathematical and physical revelations using the new techniques; for a taste:

$$\pi/4 = 1/1 - 1/3 + 1/5 - 1/7 + 1/9 - \ldots$$

Leonhard Euler (1707-1783), the most prolific mathematician of all time, "the master of us all" performed dazzling acrobatic feats that today seem so dubious that only Euler could have gotten the right answer. For example, he obtains the (correct) identity

$$e = 1/1! + 1/2! + 1/3! + 1/4! + 1/5! + 1/6! + 1/7! \ldots$$

by raising 1 + an infinitely small quantity, to an infinitely large power, and expanding using the binomial theorem! This kind of magic left many mathematicians increasingly uneasy. Zeno did have a point! Which arguments make sense and which do not?

What is the limiting value of the sequence 1, 1/2, 1/3, 1/4, 1/5, ... Obviously this gets closer and closer to zero. Cauchy's point is that this sequence limits to zero precisely because whatever your margin of error, eventually this sequence stays closer to zero than that margin.

This clears up a lot of mysteries, but adds more:

Finally, Augustin-Louis Cauchy (1789-1857) gave the key: His answer to Zeno?

I don't care if the value IS 1. You tell me how close your standard is, what error or tolerance you'll allow when I say its close to 1.

You tell me what your standard is: do you want the answer to within 1%? A zillionth? A gazillionth? Whatever your standard is,

$$1/2 + 1/4 + 1/8 + 1/16 + \ldots$$

the distance to 1 is within that tolerance.
In that sense, precisely, the limit of that sum is 1: whatever your margin of error, the sum is closer.

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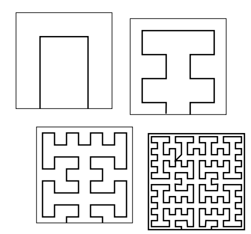

This sequence of curves is named for Peano. Is their limit a curve? Is it the whole square? Can you draw the next stage?

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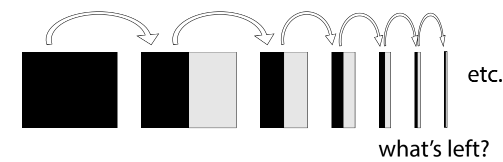
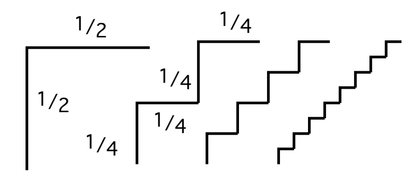
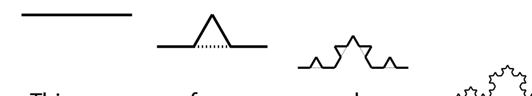
Some infinite processes make sense and some do not! Which of these have meaningful limits, in the sense of Cauchy?

cut a rectangle in half again and again and...





What is the limiting curve? Every curve in this sequence has length 1. But what is the length of their limit?



This sequence of curves approaches an infini tely long curve ! Why?

If you turn a light on and off an infinite number of times will the light be on or off?