What's the largest number

that you can write down in 100 characters read in a usual mathematical way? *

You may use digits `0 1 2 3 4 5 6 7 8 9` and symbols for arithmetic operations, such as `+` (for plus), your choice of `x` or $*$ or $\cdot$ (for times), $\uparrow$ (for exponentiation, as in $2\uparrow3 = 2^3 = 2\cdot2\cdot2 = 8$) and `!` (for taking factorials). We'll throw in some little extras like `( ) = , a b` etc.

*Here we mean the finite counting numbers, 0, its successor 1, the successor of that, 2, and so on forever. But only the first few of these are "real" in any sense. What is the largest meaningfully "real" counting number? Visualize 10 things or 1000, but how can you truly *feel* a trillion?

Check these estimates for yourself: • A billion .'s can cover the walls and floors of a normal room a 15ft $\times$ 20ft $\times$ 8ft room is $2(8\cdot15+15\cdot20+20\cdot8) = 1160$sq ft; at $(72\cdot12)^2$ dots per square foot, we have 865935360 dots – billionish for sure. — a thousand rooms are needed for a trillion dots. • There are about 10^{15} grains of sand on a beach and about 10^{16} ants in the world. • There have been $3\cdot10^{20}$ seconds since the Big Bang. • The average distance between galaxies is 10^{19} miles. • The volume of our galaxy is 10^{21} cubic miles. • There are about 24 10^{24} atoms in a glass of water, 6 10^{50} atoms in the earth, and 10^{57} in the sun. • The volume of the visible universe is 10^{68} cubic miles.

A rajju is the distance covered by a deva flying for six months at the rate of ten million miles per blink of an eye. 5 blinks per second, 30 million seconds in a year, works out to roughly 50 million blinks per six months, or 500 trillion miles., 100 light years in just six months. A palya is the length of time it takes to build a cube of lambswool ten miles high, if one strand were laid down every century. Lord Adinath came to India 100,000,000,000,000 palyas ago. assume 10 lambswool per cubic millimeter, $10\cdot(1000)^3(1000)^3(1.6)^3(10)^3 \approx 10^{23}$ centuries. Lord Adinath came to India 10^{39} years ago! In fact, a palya is 10^{12} times the length of the universe. There have been 4 10^{17} seconds since the Big Bang, so to have already created even one cube of lambswool 10 miles high, a million strands must have been laid down every second! How big are these numbers?

There are only about a googol, 10^{100}, elementary particles in the universe, and on the finest meaningful scale, the universe is only about $(10^{33}10^{27})^3 = 10^{180}$ cubic Plank units in volume.

The number of *states* of physical things can be very large. Each of our 100 boxes can be filled in with one of about 50 characters, or in other words, can be in one of about 50 states. All together there are 50^{100} possible one hundred character strings. To put it another way there are $50^{100} \approx 10^{170}$ possible states for the hundred boxes all together. This is already more than the number of elementary particles in the universe!

A computer monitor has even more states: Suppose a monitor is 1920 pixels across and 1080 high. Each pixel is colored some amount of red, green and blue — there are 2^{2^3} options for each color. All together, there are a staggering $((2^{2^3})^3)^{(1920\cdot1080)}$ possible images a monitor can display! (Though vastly most of these are just staticky, and there are far far fewer "recognizably different" images...) How large is this number?

Maybe the biggest "real counting number" is the number of states the universe could be in, which is much smaller than $(((10^{60})^3)^2)^{10^{100}}$ 10^{100} particles, each in one of at least $((10^{60})^3)^2$ ish states (momentum and position) (why) which is about $10^{360^{10^{100}}}$.On the other hand, the number of states of the universe is certainly much bigger than a googolplex , or $10^{10^{100}}$ (since each particle can certainly be in more than 10 states). Are larger counting numbers than these "real"?

What is the largest number that can be written in the 100 boxes?

Beginning with just the digits 0 1 2 3 4 5 6 7 8 9,
obviously a string of one hundred 9's, $\underbrace{\texttt{9999...9999}}_{100}$, is the biggest.

But how can this be improved?

Does the symbol + help anything?

Does the symbol * do much for us?
(Which is larger, for example 9*9*9 or 99999? More generally, if α and ω are strings of digits, which is larger, $\alpha*\omega$ or $\alpha 9\omega$? Even $\alpha 0\omega$ is bigger than $\alpha*\omega$!

If we use a ↑ character for exponentiation, we can get *much* bigger numbers.

But where should we put the ↑? For example, which of these 5-letter strings gives the largest number: 99↑99 vs. 9↑999 vs. 999↑9. How can we think about this?
(Why does 999↑9 $\approx 1000^{10} = (10^3)^{10} = 10^{30}$ have about 30 digits, 99↑99 $\approx 100^{100} = (10^2)^{100} = 10^{200}$ have about 200 digits and 9↑999 $\approx 10^{1000}$ have about 1000 digits?)

Much Bigger Numbers

With *repeated* exponentiation we can really start to go nuts!!

How big is

$$2^{3^4}\ ?$$

The order of exponentiation is always inside to outside. So this is *not* worked out as $(2^3)^4$, which is a mere $2^{3\cdot 4} = 4096$. Instead it is $2^{(3^4)}$, a much larger number, $2^{81} = 2,417,851,639,229,258,349,412,352$. But that's just a start: $2^{2^{3^4}} = 2^{2,417,851,639,229,258,349,412,352}$ is much bigger still! *

So that we can write these towers out, we're going to switch to up-arrows ↑ for exponentiation, writing for example 2↑2↑3↑4 for $2^{2^{3^4}}$.

How big is 3↑3↑3↑3= $3^{3^{3^3}}$? Of course $3^3 = 27$, so $3\uparrow 3\uparrow 3 = 3^{27}$. Since 3^2 is about 10, more or less, $3^{27} \approx 10^{27/2}$, a thirteen digit number (7,625,597,484,987 to be precise).

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*We can estimate how many digits $2^{2^{3^4}} \approx 2^{2\cdot 10^{24}}$ has: 2^{10} is about 10^3, so swapping every 10 powers of 2 for 3 powers of 10, this is about a 800,000,000,000,000,000,000,000 digit number! That's are about $1\frac{1}{4}$ moles worth of *digits*, as many atoms in $1\frac{1}{4}$ kilograms of hydrogen!

So $3^{3^{3^3}}$ is about 3 raised to a thirteen digit number, or 10 raised to half that (a three trillion-ish digit number). In other words, 3↑3↑3↑3 has about 10^{13} digits!! The number 3↑3↑3↑3↑3 has about *that* number of digits, whatever that might mean!

The Knuth Arrow Notation

But that's just baby stuff!! *We've hardly even started!!*

The Knuth Arrow Notation lets us lift off even further. To see how this works, remember that:

Multiplication is just repeated *addition*: $3 \times 4 = 3 + 3 + 3 + 3$ (four 3's are added)*

Exponentiation is just repeated *multiplication*: $3 \uparrow 4 = 3 \times 3 \times 3 \times 3$ (four 3's are multiplied)

What shall we call repeated *exponentiation*, such as $3 \uparrow 3 \uparrow 3 \uparrow 3$? Knuth uses the symbol $3 \uparrow\uparrow 4$. That is, four 3's are "exponentiated" Remember that $3 \uparrow 3 \uparrow 3 \uparrow 3 = 3 \uparrow (3 \uparrow (3 \uparrow 3)) = 3^{3^{27}}$ a three-trillion digit number.

$3 \uparrow\uparrow\uparrow 3 = 3 \uparrow\uparrow 3 \uparrow\uparrow 3$ is pretty bad: The order of operations works from the right, just as with ordinary exponentiation, so we work out $3 \uparrow\uparrow (3 \uparrow\uparrow 3)$. We've already worked out that $3 \uparrow\uparrow 3$ is about seven and a half trillion, so we have that $3 \uparrow\uparrow\uparrow 3$ is about $3 \uparrow\uparrow$ (seven and a half trillion). This is disturbing:

$$3^{3^{3^{3^{3^{\cdot^{\cdot^{\cdot^{3}}}}}}}}$$

a tower of 3's seven and a half trillion high!!

But then $3 \uparrow\uparrow\uparrow 4 = 3 \uparrow\uparrow (3 \uparrow\uparrow\uparrow 3)$, an exponential tower of 3's that is $(3 \uparrow\uparrow\uparrow 3)$ high!! Care to think about what $3 \uparrow\uparrow\uparrow\uparrow 3 = 3 \uparrow\uparrow\uparrow (3 \uparrow\uparrow\uparrow 3)$ is?

Weirdly, by the way, $2 + 2 = 2 \times 2 = 2 \uparrow 2 = 2 \uparrow\uparrow 2 = 2 \uparrow\uparrow\uparrow 2 = \ldots = 4$ since 2×2 is two 2's added, $2 \uparrow 2$ is two 2's multiplied, $2 \uparrow\uparrow 2$ is two 2's "exponentiated", etc, etc.

And for any n, why does $n \times 1 = n \uparrow 1 = n \uparrow\uparrow 1 = ... = n$?

If we really want to compare two numbers, like $5 \uparrow\uparrow\uparrow 5$ and $6 \uparrow\uparrow\uparrow 4$,‡ we have to boil them

*And addition is just repeated *succession*: $a + b$ is a succeeded b times.
‡Can you work out which is bigger?

down to something we understand — all the way to comparing tally marks if we have to! Here's a rule for reducing these out one step at a time, though this will take a long long while to expand out all the way to tallys. Let's start with an example:

$3 \uparrow\uparrow 5 = 3 \uparrow 3 \uparrow 3 \uparrow 3 \uparrow 3$. Regrouping, we have that this equals $3 \uparrow (3 \uparrow\uparrow 4)$. Once we've worked out $(3 \uparrow\uparrow 4)$, we'll have our answer. But $3 \uparrow\uparrow 4 = 3 \uparrow (3 \uparrow\uparrow 3)$, which we can work out once we have $3 \uparrow\uparrow 3 = 3 \uparrow (3 \uparrow\uparrow 2)$, which we can work out knowing that $(3 \uparrow\uparrow 2) = 3 \uparrow (3 \uparrow\uparrow 1)$. Here we have something we already knew: why is $3 \uparrow\uparrow 1 = 3$?

In the same way, we can reduce $3 \uparrow\uparrow\uparrow 5 = 3 \uparrow\uparrow (3 \uparrow\uparrow\uparrow 4)$. If we have worked out what $3 \uparrow\uparrow\uparrow 4$ is, which we can in the same way, then $3 \uparrow\uparrow$ (that large number) $= 3 \uparrow (3 \uparrow\uparrow$ (that large number)–1).

The point is, that we can alway boil these down into smaller pieces, and we have the general formula that

$$a \underbrace{\uparrow \ldots \uparrow}_{n \ \uparrow\text{'s}} b = a \underbrace{\uparrow \ldots \uparrow}_{(n-1) \ \uparrow\text{'s}} \ (a \underbrace{\uparrow \ldots \uparrow}_{n \ \uparrow\text{'s}} (b-1))$$

(as long as $n > 1$ and $b > 1$).

Eventually we boil down to $n = 1$ or $b = 1$: $a \uparrow b = a^b$ and $a \underbrace{\uparrow \ldots \uparrow}_{n \ \uparrow\text{'s}} 1 = a$.

Recursive functions

If we use function notation we can generalize this and produce some really enormous numbers in just a few characters! Let's introduce letters `a`, `b`, etc, parentheses `( )`, equals* signs `=` and, to keep things organized, commas and semicolons `,  ;` too.

For our first example, let's define a function $f(a) = a * f(a-1)$. We could work out $f(10)$ as $f(10) = 10 * f(9)$. We can work out $f(9)$ as $9 * f(8)$, and so on boiling things down. In this way $f(10) = 10 * 9 * f(8) = 10 * 9 * 8 * \cdots =$
But where does this stop? We had better define a base case, such as $f(1) = 1$. You probably recognize what function this `f` defines:

```
f(1)= 1; f(a)=a * f(a-1)
```

With just a few more characters, we can define a `R`*epeated* factorial function.
Since $a \underbrace{!! \ldots !}_{n} = (a!) \underbrace{! \ldots !}_{n-1}$, lets define:

*Really, the "defined to be equal sign" `:=` is better, but we don't want to argue about how many characters that is.

```
f(1)= 1; f(a)=a * f(a-1);     R(a,1)=f(a); R(a,n) = R(f(a,1),n-1)
```

Then $R(a,n)$ represents $a\underbrace{!!\ldots!}_{n}$ *(!)*

So for example, $R(9,3) = 9!!! = (9!)!! = (362880)!! = (\text{some staggering number})!$ *(!!)* †

In our hundred boxes, we have plenty of room to write down some huge numbers: We used 33 characters to define our function, and we could use the rest to define input for our function, such as $\texttt{R(9,}\underbrace{\texttt{9...9}}_{\texttt{62 9's}}\texttt{)}$.

But we can make far, far larger numbers with only a few characters. How big are `R(9,R(9,9))` or even `R(9,R(9,R(9,9))`???

Can we do even more??

Here's a way to define a function `K` that gives $a\underbrace{\uparrow\ldots\uparrow}_{n}b$, with Knuth arrows.

Remember we have a way to simplify this as $a\underbrace{\uparrow\ldots\uparrow}_{n-1}\ (a\underbrace{\uparrow\ldots\uparrow}_{n}(b-1))$. So let's define

```
K(a,1,b)=a↑b;
K(a,n,1)=a;
K(a,n,b)=K(a,n-1,K(a,n,b-1));
```

Check that `K(a,n,b)` $= a\underbrace{\uparrow\ldots\uparrow}_{n}b$.

NOW define

```
G(1) = K(3,4,3);
G(n) = K(3,G(n-1),3);
```

So $G(1) = K(3,4,3) = 3 \uparrow\uparrow\uparrow\uparrow 3$, an unimaginably large, but well-defined number.‡
Next, $G(2) = 3\underbrace{\uparrow\ldots\uparrow}_{G(1)}3$, which has $G(1)$ *Knuth arrows*!!!! And $G(3)$ has $G(2)$'s worth of arrows! And this just goes on!

†How many Knuth arrows does repeated exponentiation compare to?
In general, for any $n \geq 2$, $n!$ is less than n^n. On the other hand, $n!!$ is much greater than n^n, since $n!! = (n!)(n!-1)...(n!-n)\ldots$ — the first n terms are each bigger than n. For example $9!! = (9!)! < (9!)^{(9!)} < (9^9)^{(9^9)} = 9^{9\cdot 9^9} = 9^{9^{9+1}} \approx 9 \uparrow\uparrow 3$. On the other hand $9!! > (9!)^{(9!)} > 9 \uparrow 9 = 9 \uparrow\uparrow 2$. Similarly, verify that $9 \uparrow\uparrow 3 < 9!!! < 9 \uparrow\uparrow 4$. What's the pattern? Prove it holds forever.

‡$G(1) = K(3,4,3) = 3 \uparrow\uparrow\uparrow\uparrow 3 = 3 \uparrow\uparrow\uparrow (3 \uparrow\uparrow\uparrow 3)$. Recall $3 \uparrow\uparrow\uparrow 3 = 3 \uparrow\uparrow (3^{3^3})$, an exponential tower 3^{27} 3's tall; $G(1)$ is this many repetitions of the $\uparrow\uparrow$ operation!

The famous Graham's Number is $G(64)$: This fits in just under 100 characters:

```
K(a,1,b)=a↑b;
K(a,n,1)=a;
K(a,n,b)=K(a,n-1,K(a,n,b-1));
G(1)=K(3,4,3);
G(n)=K(3,G(n-1),3);
G(64)
```

With another ten or twenty characters how much bigger can you go?

What is the largest number you can write down, one character per box below, read in a usual mathematical way?

You may use digits 0 1 2 3 4 5 6 7 8 9 and symbols for arithmetic operations, such as + (for plus), your choice of x or $*$ or $\cdot$ (for times), $\uparrow$ (for exponentiation, as in 2$\uparrow$3 $= 2^3 = 2{\cdot}2{\cdot}2 = 8$) and ! (for taking factorials), and some little extras like ( ) = , a b etc.

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