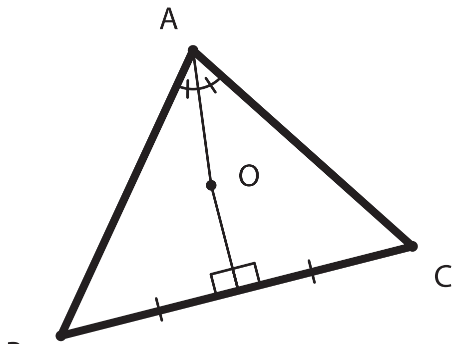
A “proof” that every triangle is equilateral.

It is remarkably difficult to spot the error—this will be your homework for the year. If you make a careful sketch in Geometer’s Sketchpad, you will be able to discover where the flaw is. Hint: all the triangles we will claim are congruent really are—that’s not the problem.

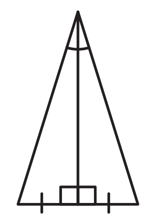


Clearly, an equilateral triangle?

Given any triangle ABC, we shall prove that AB is congruent to AC. Consequently, since we may label the vertices however we please and the proof will still “work”, this implies that all three sides are congruent and the triangle is equilateral!

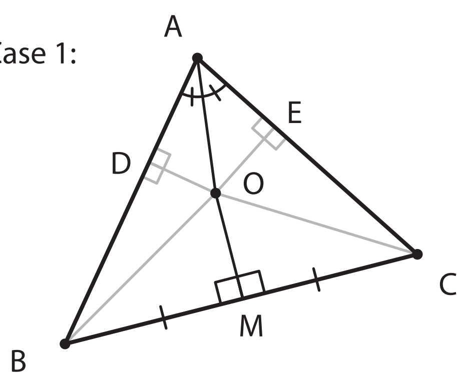
Begin by constructing the angle bisector of angle BAC, and the perpendicular bisector of segment BC. Suppose for contradiction that triangle ABC is not isosceles. Then these lines are not parallel and meet at some point O.

Clearly O cannot be on segments AB or AC (since O lies on the bisector of the angle between them). And if O lies on segment BC, the angle bisector is then be the perpendicular bisector of BC, the triangle would be isosceles (AAS), and we have proven AB is congruent to AC as promised..

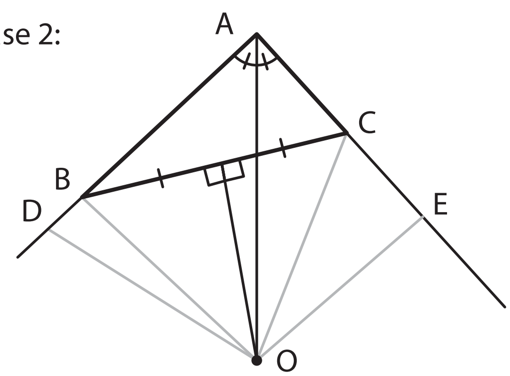


We still have two cases to consider: The point O is inside the triangle ABC, or the point O is outside the triangle B.

Case 1:



Case 2:



Let M be the midpoint of BC.
From O drop the perpendiculars to sides AB and AC, to points D and E, and draw segments BO and CO.

Now, triangles BMO and CMO are congruent, by SAS, so segments BO and CO are congruent.

Triangles DOA and EOA are congruent by AAS, so segments DO and EO are congruent, as are segments AD and AE.

Triangles DOB and EOC are both right triangles, and have congruent hypotenuses and a congruent leg. By the Pythagorean theorem the other legs, segments BD and CE are congruent as well, and so by SSS are congruent triangles.

Segment AB is equal to AD and DB. Segment AC is equal to AE and EC. Since segment AD is congruent to segment AE, and segment DB is congruent to segment EC, we have proved that segment AB is congruent to AC.

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From O drop the perpendiculars to lines AB and AC, to points D and E, and draw segments BO and CO.

Now, triangles BMO and CMO are congruent, by SAS, so segments BO and CO are congruent.

Triangles DOA and EOA are congruent by AAS, so segments DO and EO are congruent, as are segments AD and AE.

Triangles DOB and EOC are both right triangles, and have congruent hypotenuses and a congruent leg. By the Pythagorean theorem they have the other leg congruent as well, and so by SSS are congruent triangles.

Segment AD is equal to AB and BD. Segment AE is equal to AC and CE. Since segment AD is congruent to segment AE, and segment DB is congruent to segment EC, we have proved that segment AB is congruent to AC.